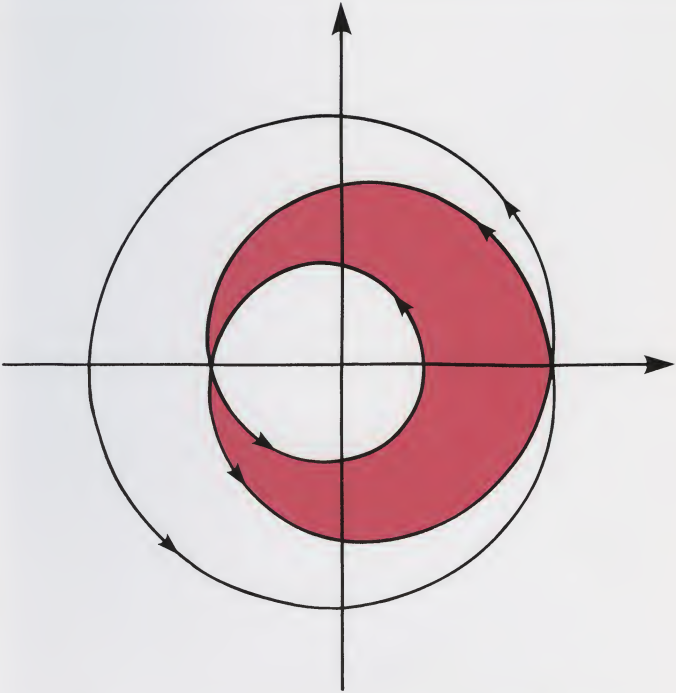
M337 Unit C3
Mathematics: A Third Level Course

The Open University

COMPLEX ANALYSIS

UNIT C3 ANALYTIC CONTINUATION



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Prepared by the Course Team

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Before working through this text, make sure that you have read the *Course Guide* for M337 Complex Analysis.

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1.4

CONTENTS

INTRODUCTION

By now, you may have reached the conclusion that complex analysis is largely due to Cauchy, since so many theorems bear his name. It is certainly true that many of the foundations of the subject were built by Cauchy in papers written between 1820 and 1850, and it was in this period that mathematicians became aware of the need for great care when dealing with all limiting processes; for example, the need to check that any series used in a proof is in fact convergent.

In this unit we reach various topics of complex analysis which were developed more or less after Cauchy and are mainly associated with Riemann and Weierstrass. The common theme running through these topics is *analytic continuation*, the process of extending or continuing the domain of a given analytic function while preserving its analyticity.

We introduce this notion in Section 1 and make some applications to improper integrals. In Section 2, we describe analytic continuation by Taylor series and also sketch the construction of a simple *Riemann surface*, an ingenious device for representing the domain of a so-called 'multi-valued function' so that it can be interpreted as an ordinary 'one-valued' function.

In Section 3, we introduce the idea of *uniform convergence* of sequences and of series of functions. This idea, known to Cauchy, was exploited by Weierstrass to define further analytic functions as limits of various elementary analytic functions. In particular, we introduce the zeta function

$$\zeta(z) = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots,$$

and prove that it is analytic on the open half-plane $\{z : \operatorname{Re} z > 1\}$.

Section 4 is devoted to the basic properties of the gamma function. This is an analytic version of the factorial function $f(n) = n!$ and is defined by the improper integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt.$$

Finally in Section 5, we briefly describe Riemann's remarkable scheme for applying complex analysis to number theory, which led eventually to a proof of the Prime Number Theorem on the distribution of primes, and bequeathed to future generations an intriguing problem about the zeta function which is still unsolved.

Study guide

This unit contains some relatively advanced topics in complex analysis (for example, analytic continuation and uniform convergence), some of which you may find difficult to assimilate, especially on a first reading. You should note that uniform convergence is an important general concept which occurs in all branches of analysis.

The material in Section 5, Subsection 4.4 and the last part of Subsection 2.2 is intended for reading only.

Subsection 1.2 contains the audio tape, which introduces further methods for evaluating certain improper integrals.

1 WHAT IS ANALYTIC CONTINUATION?

After working through this section, you should be able to:

(a) determine *direct analytic continuations* of certain analytic functions;

(b) evaluate certain improper integrals involving logarithms or non-integer powers.

1.1 Direct analytic continuation

An analytic function f is usually specified by giving a rule $f(z)$, which is an expression involving the variable z, and a region $\mathcal{R}$ on which f is analytic. For example, the sum function f of the power series $1 + z + z^2 + \cdots$ is

$$f(z) = 1 + z + z^2 + \cdots \qquad (|z| < 1) \tag{1.1}$$

and f is analytic on $D = \{z : |z| < 1\}$, the disc of convergence of the power series. Since this power series is a geometric series, we could equally well have defined the function f by using the formula for the sum of this series, that is,

$$f(z) = \frac{1}{1-z} \qquad (|z| < 1).$$

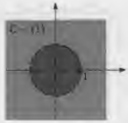


Figure 1.1

But the expression $1/(1-z)$ is defined for all z in the larger region $\mathbb{C} - \{1\}$ (see Figure 1.1). Thus f is the restriction to D of the analytic function

$$g(z) = \frac{1}{1-z} \qquad (z \in \mathbb{C} - \{1\}). \tag{1.2}$$

Put another way, g is an **analytic extension** of f from D to $\mathbb{C} - \{1\}$, that is, g is a function which is analytic on the larger set $\mathbb{C} - \{1\}$, and which agrees with f on the unit disc D.

This notion of an analytic extension also arose in *Unit B4*, in connection with the idea of a removable singularity. For example, the function

$$f(z) = \frac{\sin z}{z} \qquad (z \neq 0) \tag{1.3}$$

is analytic on $\mathbb{C} - \{0\}$ and it has a removable singularity at 0 because the function

Unit B4, Frame 1

$$g(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots = \begin{cases} \dfrac{\sin z}{z}, & z \neq 0, \\ 1, & z = 0, \end{cases}$$

is analytic on $\mathbb{C}$. Thus g is an analytic extension of f from $\mathbb{C} - \{0\}$ to $\mathbb{C}$.

Example 1.1

Determine an analytic extension of the function

$$f(z) = \sum_{n=0}^{\infty} (-1)^{n+1} (z-2)^n \qquad (|z-2| < 1).$$

Solution

Since $\sum_{n=0}^{\infty} (-1)^{n+1} (z-2)^n = -1 + (z-2) - (z-2)^2 + \cdots$ is a geometric series with sum

$$\frac{-1}{1 + (z-2)} = \frac{1}{1-z},$$

we deduce that the function

$$g(z) = \frac{1}{1-z} \qquad (z \in \mathbb{C} - \{1\})$$

is an analytic extension of f to $\mathbb{C} - \{1\}$. ■

Problem 1.1

Determine an analytic extension of each of the following functions.

(a) $f(z) = \sum_{n=0}^{\infty} (2z)^n \qquad (|z| < \frac{1}{2})$

(b) $f(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(z-1)^n \qquad (|z-1| < 1)$

For a given analytic function f, it is natural to seek the largest region to which f can be extended analytically. For example, the function f defined by Equation (1.1) can be analytically extended to $\mathbb{C} - \{1\}$ by Equation (1.2), but no further because the function $g(z) = 1/(1-z)$ has a pole at 1. Similarly, the function f defined by Equation (1.3) can be analytically extended to the whole of $\mathbb{C}$, and this is clearly the largest possible such region.

Note that g must be the *unique* analytic extension of f to $\mathbb{C} - \{1\}$, by the Uniqueness Theorem (*Unit B3*, Theorem 5.5).

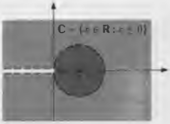
For some functions f, however, there is no unique largest region in $\mathbb{C}$ to which f can be extended analytically. For example, in Problem 1.1(b) you saw that the function

$$f(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(z-1)^n \qquad (|z-1| < 1)$$

has the analytic extension

$$g(z) = \operatorname{Log} z \qquad (z \in \mathbb{C} - \{x \in \mathbb{R} : x \le 0\})$$

(see Figure 1.2).



Now it is certainly not possible to extend this function g to a region which is larger than $\mathbb{C} - \{x \in \mathbb{R} : x \le 0\}$, because the function Log is not analytic at any point of the negative real axis. So it might appear that we have found the largest region to which f can be analytically extended. However, consider the function

$$h(z) = \operatorname{Log}_{3\pi/2}(z) \qquad (z \in \mathbb{C}_{3\pi/2}).$$

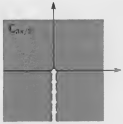


Figure 1.3

Here $h(z) = \log_e |z| + i \operatorname{Arg}_{3\pi/2}(z)$ and $\mathbb{C}_{3\pi/2}$ is the cut plane illustrated in Figure 1.3.

Because

$$\operatorname{Arg}_{3\pi/2}(z) = \operatorname{Arg} z, \qquad \text{for } \operatorname{Re} z > 0,$$

we have

$$h(z) = g(z), \qquad \text{for } \operatorname{Re} z > 0,$$

and so

$$h(z) = f(z), \qquad \text{for } |z-1| < 1.$$

The functions Log_ϕ and Arg_ϕ were introduced in *Unit C2*, Section 1. In particular, recall that

$$\operatorname{Log}_\pi = \operatorname{Log}, \quad \operatorname{Arg}_\pi = \operatorname{Arg}.$$

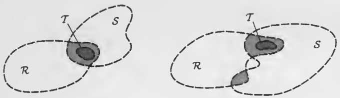
Thus h is also an analytic extension of f, but it extends f to the region $\mathbb{C}_{3\pi/2}$ which is neither smaller nor larger than $\mathbb{C} - \{x \in \mathbb{R} : x \le 0\}$. Thus, it does not always make sense to seek *the* largest region to which a given analytic function can be extended. Instead we introduce a related idea called *analytic continuation*, and discuss to what extent a given analytic function can be analytically continued.

Definitions Let f and g be analytic functions whose domains are the regions $\mathcal{R}$ and $\mathcal{S}$, respectively. Then f and g are **direct analytic continuations** of each other if there is a region $\mathcal{T} \subseteq \mathcal{R} \cap \mathcal{S}$ such that

$$f(z) = g(z), \qquad \text{for } z \in \mathcal{T}.$$

(See Figure 1.4.)

We also say that g is a **direct analytic continuation of f from $\mathcal{R}$ to $\mathcal{S}$** and vice versa.



(a) $\mathcal{R} \cap \mathcal{S}$ is a region (b) $\mathcal{R} \cap \mathcal{S}$ is not a region

Figure 1.4

Remarks

1 The word 'direct' appears in this definition because there is a notion of indirect analytic continuation which we discuss in Section 2. Often we speak simply of 'analytic continuation'.

2 Notice that, by the Uniqueness Theorem, there can be at most one function g which is analytic on $\mathcal{S}$ and agrees with f on the region $\mathcal{T}$. This is often referred to as the **uniqueness of analytic continuation**.

3 Note that we do not insist in this definition that f and g are equal throughout $\mathcal{R} \cap \mathcal{S}$. However, if $\mathcal{R} \cap \mathcal{S}$ is a region (as in Figure 1.4(a)), then the equality of f and g on $\mathcal{T}$ implies their equality on $\mathcal{R} \cap \mathcal{S}$, by the Uniqueness Theorem.

4 The above definitions, which are symmetric with respect to f and g, apply in the special case for which $\mathcal{S} \subseteq \mathcal{R}$. Then only the analytic continuation of g from $\mathcal{S}$ to $\mathcal{R}$ is of interest.

Example 1.2

Prove that each of the following pairs of analytic functions f and g are direct analytic continuations of each other.

(a) $f(z) = \sum_{n=0}^{\infty} z^n \quad (|z| < 1)$ and $g(z) = \dfrac{1}{1-z} \quad (z \in \mathbb{C} - \{1\})$

(b) $f(z) = \operatorname{Log}_{\pi}(z) \quad (z \in \mathbb{C}_{\pi})$ and $g(z) = \operatorname{Log}_{3\pi/2}(z) \quad (z \in \mathbb{C}_{3\pi/2})$

Solution

(a) Here $\mathcal{R} = \{z : |z| < 1\}$ and $\mathcal{S} = \mathbb{C} - \{1\}$. Since f and g agree on the region $\mathcal{T} = \{z : |z| < 1\} \subseteq \mathcal{R} \cap \mathcal{S}$, we deduce that f and g are direct analytic continuations of each other.

(b) Here $\mathcal{R} = \mathbb{C}_{\pi}$ and $\mathcal{S} = \mathbb{C}_{3\pi/2}$. Since f and g agree on the region $\mathcal{T} = \{z : \operatorname{Re} z > 0\} \subseteq \mathcal{R} \cap \mathcal{S}$, we deduce that f and g are direct analytic continuations of each other. ■

Remark In part (a), $\mathcal{R} \subseteq \mathcal{S}$ and f and g agree on the region $\mathcal{R} \cap \mathcal{S} = \mathcal{R}$. In such cases, $\mathcal{T} = \mathcal{R}$ is the natural choice for $\mathcal{T}$.

Problem 1.2

Prove that each of the following pairs of analytic functions f and g are direct analytic continuations of each other.

(a) $f(z) = \sum_{n=1}^{\infty} nz^{n-1} \quad (|z| < 1)$ and $g(z) = \dfrac{1}{(1-z)^2} \quad (z \in \mathbb{C} - \{1\})$

(b) $f(z) = \operatorname{Log}_{2\pi}(z) \quad (z \in \mathbb{C}_{2\pi})$ and $g(z) = \operatorname{Log}_{3\pi/2}(z) \quad (z \in \mathbb{C}_{3\pi/2})$

Often, we are given one function f analytic on a region $\mathcal{R}$ and are required to find a direct analytic continuation of f from $\mathcal{R}$ to some overlapping region $\mathcal{S}$. The following problem gives you an opportunity to try this.

Problem 1.3

Use Example 1.2(b) to find a direct analytic continuation of the following function f from its domain to another region.

$$f(z) = \sqrt{z} \qquad (z \in \mathbb{C}_\pi)$$

(*Hint*: Remember that $\sqrt{z} = \exp(\frac{1}{2} \operatorname{Log} z)$, for $z \neq 0$.)

1.2 More improper integrals (audio-tape)

In this subsection we show how the Residue Theorem can be combined with a simple direct analytic continuation to evaluate improper integrals of the form:

$$\int_0^\infty \frac{p(t)}{q(t)} \log_e t\,dt, \quad \int_0^\infty \frac{p(t)}{q(t)} t^a\,dt, \quad \int_0^\infty \frac{p(t)}{q(t)}\,dt,$$

where $0 < a < 1$ and p, q are polynomial functions such that the degree of q exceeds that of p by at least two and any poles of p/q on the non-negative real axis are simple. Such integrals may be 'improper at 0' and so we must define what such an integral means.

Definition Let a function f be continuous on the interval $]0, \infty[$. Then

$$\int_0^\infty f(t)\,dt = \lim_{\varepsilon \to 0} \int_\varepsilon^1 f(t)\,dt + \lim_{r \to \infty} \int_1^r f(t)\,dt,$$

provided that both these limits exist. (See Figure 1.5.)

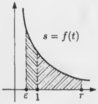


Figure 1.5

Remarks

1 We often represent this limiting process by writing

$$\int_0^\infty f(t)\,dt = \lim_{\substack{r \to \infty \\ \varepsilon \to 0}} \int_\varepsilon^r f(t)\,dt.$$

2 The limit of integration 1 is purely a convenient choice.

There are two problems associated with integrals of the above forms. The first is that such an integral is only taken along the non-negative real axis, so the standard semicircular contour used in *Unit C1*, Section 3 may not be appropriate. The other problem is the expressions $\log_e t$ and t^a, which suggest that the principal logarithm function Log will be needed. Unfortunately this function is analytic only on the cut plane $\mathbb{C} - \{x \in \mathbb{R} : x \le 0\}$, which does not include all of our standard semicircular contour. As you will see, the solution is to use various analytic continations of Log.

Before starting the tape, you should attempt the following problem, whose result you will need.

Problem 1.4

(a) Show that

$$\int_1^r \frac{1}{t}\,dt \le \int_1^r \frac{1}{\sqrt{t}}\,dt, \qquad \text{for } r > 1,$$

and deduce that

$$\log_e r \le 2\sqrt{r}, \qquad \text{for } r > 1. \qquad (*)$$

(b) Use Inequality $(*)$ to prove that

(i) $\dfrac{\log_e r}{r} \to 0$ as $r \to \infty$;

(ii) $\varepsilon \log_e \varepsilon \to 0$ as $\varepsilon \to 0$.

NOW START THE TAPE.

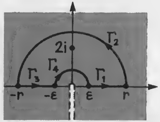
1. Evaluate $\displaystyle\int_0^\infty \frac{\log_e t}{t^2+4}\,dt$

$= \displaystyle\lim_{\substack{r\to\infty\\ \varepsilon\to 0}} \int_\varepsilon^r \frac{\log_e t}{t^2+4}\,dt$

(a) Consider the contour integral

$$I = \int_\Gamma \frac{\mathrm{Log}_{3\pi/2}(z)}{z^2+4}\,dz$$

$\mathrm{Log}_{3\pi/2}(z) = \log_e|z| + i\,\mathrm{Arg}_{3\pi/2}(z)$



$\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$

(b) Use the Residue Theorem

$$I = 2\pi i \times (\text{residue at } 2i)$$
$$= 2\pi i \times \frac{\mathrm{Log}_{3\pi/2}(2i)}{2\times 2i}$$
$$= \frac{\pi}{2}\left(\log_e 2 + i\frac{\pi}{2}\right).$$

g/h Rule:
$g(z) = \mathrm{Log}_{3\pi/2}(z)$
$h(z) = z^2+4,\ h'(z) = 2z$

$\mathrm{Log}_{3\pi/2}(2i) = \log_e|2i| + i\,\mathrm{Arg}_{3\pi/2}(2i)$
$= \log_e 2 + i\,\pi/2$

(c) Split up the integral

$$I = \int_{\Gamma_1}\cdots dz + \int_{\Gamma_2}\cdots dz + \int_{\Gamma_3}\cdots dz + \int_{\Gamma_4}\cdots dz, \qquad (*)$$

where

$$\int_{\Gamma_1}\cdots dz = \int_\varepsilon^r \frac{\log_e t}{t^2+4}\,dt$$

and

$$\int_{\Gamma_3}\cdots dz = \int_{-r}^{-\varepsilon} \frac{\log_e|t| + i\pi}{t^2+4}\,dt = \int_\varepsilon^r \frac{\log_e t}{t^2+4}\,dt + i\pi\int_\varepsilon^r \frac{1}{t^2+4}\,dt.$$

$\displaystyle\int_{-r}^{-\varepsilon}\cdots dt = \int_\varepsilon^r \cdots dt$

Integrand is even.

(d) Estimate the integrals along Γ_2 and Γ_4

On Γ_2, $|z| = r$, so $|z^2+4| \geq r^2 - 4$, and
$$|\mathrm{Log}_{3\pi/2}(z)| = |\log_e|z| + i\,\mathrm{Arg}_{3\pi/2}(z)|$$
$$\leq \log_e r + \pi.$$

Triangle Inequality

Thus

$$\left|\int_{\Gamma_2}\frac{\mathrm{Log}_{3\pi/2}(z)}{z^2+4}\,dz\right| \leq \frac{\log_e r + \pi}{r^2-4}\times \pi r.$$

$r > 2$

Estimation Theorem

Similarly

$$\left|\int_{\Gamma_4}\frac{\mathrm{Log}_{3\pi/2}(z)}{z^2+4}\,dz\right| \leq \frac{\pi - \log_e\varepsilon}{4-\varepsilon^2}\times \pi\varepsilon.$$

$0 < \varepsilon < 1$

(e) Let $r\to\infty$ and $\varepsilon\to 0$ in (*)

$$\int_{\Gamma_1}\cdots dz + \int_{\Gamma_3}\cdots dz + \int_{\Gamma_2}\cdots dz + \int_{\Gamma_4}\cdots dz = \frac{\pi}{2}\left(\log_e 2 + i\frac{\pi}{2}\right)$$

$= 2\displaystyle\int_\varepsilon^r \frac{\log_e t}{t^2+4}\,dt + i\pi\int_\varepsilon^r \frac{1}{t^2+4}\,dt$

$\displaystyle\lim_{r\to\infty}\int_{\Gamma_2}\cdots dz = 0;\ \lim_{\varepsilon\to 0}\int_{\Gamma_4}\cdots dz = 0$
by (d)

$(\log_e r)/r \to 0$ as $r\to\infty$
$\varepsilon\log_e\varepsilon \to 0$ as $\varepsilon\to 0$

Equating real (and imaginary) parts:

$$2\int_0^\infty \frac{\log_e t}{t^2+4}\,dt = \frac{\pi}{2}\log_e 2 \quad \left(\text{and } \pi\int_0^\infty \frac{1}{t^2+4}\,dt = \frac{\pi^2}{4}\right),$$

giving

$$\int_0^\infty \frac{\log_e t}{t^2+4}\,dt = \frac{\pi}{4}\log_e 2 \quad \left(\text{and } \int_0^\infty \frac{1}{t^2+4}\,dt = \frac{\pi}{4}\right).$$

Now try Problem 1.5.

2. Evaluate $\displaystyle\int_0^\infty \frac{t^{-1/2}}{t^2+1}\,dt$

$= \displaystyle\lim_{\substack{r\to\infty\\ \varepsilon\to 0}} \int_\varepsilon^r \frac{t^{-1/2}}{t^2+1}\,dt$

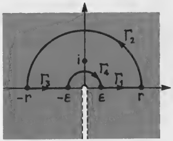
$t^{-1/2} = \exp(-\frac{1}{2}\log_e t), t>0$

(a) Consider the contour integral

$$I = \int_\Gamma \frac{\exp(-\frac{1}{2}\operatorname{Log}_{3\pi/2}(z))}{z^2+1}\,dz,$$

where

$$\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4.$$



(b) Use the Residue Theorem

$$I = 2\pi i \times (\text{residue at } i)$$

$$= 2\pi i \times \left(\frac{\exp(-\frac{1}{2}\operatorname{Log}_{3\pi/2}(i))}{2i}\right)$$

$$= \pi \exp(-i\pi/4).$$

g/h Rule:
$g(z) = \exp(-\frac{1}{2}\operatorname{Log}_{3\pi/2}(z))$
$h(z) = z^2+1, h'(z) = 2z$

$\operatorname{Log}_{3\pi/2}(i) = \log_e|i| + i\operatorname{Arg}_{3\pi/2}(i) = i\pi/2$

(c) Split up the integral

$$I = \int_{\Gamma_1}\cdots dz + \int_{\Gamma_2}\cdots dz + \int_{\Gamma_3}\cdots dz + \int_{\Gamma_4}\cdots dz \quad \begin{matrix} = \pi e^{-i\pi/4} \\ \text{by (b)} \end{matrix} \quad (*)$$

where

$$\int_{\Gamma_1}\cdots dz = \int_\varepsilon^r \frac{\exp(-\frac{1}{2}\log_e t)}{t^2+1}\,dt = \int_\varepsilon^r \frac{t^{-1/2}}{t^2+1}\,dt$$

and

$$\int_{\Gamma_3}\cdots dz = \int_{-r}^{-\varepsilon} \frac{\exp(-\frac{1}{2}(\log_e|t|+i\pi))}{t^2+1}\,dt = e^{-i\pi/2}\int_{-r}^{-\varepsilon} \frac{|t|^{-1/2}}{t^2+1}\,dt.$$

Integrand is even.

(d) Estimate the integrals along Γ_2 and Γ_4

Now

$$|\exp(-\tfrac{1}{2}\operatorname{Log}_{3\pi/2}(z))| = \exp(-\tfrac{1}{2}\log_e|z|) = |z|^{-1/2}$$

and

$$|z^2+1| \geq \begin{cases} r^2-1, & \text{on } \Gamma_2, \\ 1-\varepsilon^2, & \text{on } \Gamma_4. \end{cases}$$

Triangle Inequality

Hence

$$\left|\int_{\Gamma_2} \frac{\exp(-\frac{1}{2}\operatorname{Log}_{3\pi/2}(z))}{z^2+1}\,dz\right| \leq \frac{r^{-1/2}}{r^2-1}\times \pi r = \frac{\pi r^{1/2}}{r^2-1} \quad r>1$$

Estimation Theorem

and

$$\left|\int_{\Gamma_4} \frac{\exp(-\frac{1}{2}\operatorname{Log}_{3\pi/2}(z))}{z^2+1}\,dz\right| \leq \frac{\varepsilon^{-1/2}}{1-\varepsilon^2}\times \pi\varepsilon = \frac{\pi\varepsilon^{1/2}}{1-\varepsilon^2}. \quad 0<\varepsilon<1$$

(e) Let $r\to\infty$ and $\varepsilon\to 0$ in (*)

$$\int_\varepsilon^r \frac{t^{-1/2}}{t^2+1}\,dt + e^{-i\pi/2}\int_\varepsilon^r \frac{|t|^{-1/2}}{t^2+1}\,dt + \int_{\Gamma_2}\cdots dz + \int_{\Gamma_4}\cdots dz = \pi e^{-i\pi/4}$$

$\left(1+e^{-i\pi/2}\right)\displaystyle\int_\varepsilon^r \frac{t^{-1/2}}{t^2+1}\,dt$

$\displaystyle\lim_{r\to\infty}\int_{\Gamma_2}\cdots dz = 0;\ \lim_{\varepsilon\to 0}\int_{\Gamma_4}\cdots dz = 0$ by (d)

Hence

$$\int_0^\infty \frac{t^{-1/2}}{t^2+1}\,dt = \lim_{\substack{r\to\infty\\ \varepsilon\to 0}} \int_\varepsilon^r \frac{t^{-1/2}}{t^2+1}\,dt$$

$$= \frac{\pi e^{-i\pi/4}}{1+e^{-i\pi/2}} = \frac{\pi}{\sqrt{2}}.$$

Now try Problem 1.6.

Problem 1.5

Use the method in Frame 1 to show that

$$\int_0^\infty \frac{\log_e t}{t^2+1}\,dt = 0.$$

Problem 1.6

Use the method in Frame 2 to show that if $0 < a < 1$, then

$$\int_0^\infty \frac{t^{a-1}}{t^2+1}\,dt = \frac{\pi}{2\sin(a\pi/2)}.$$

In the audio tape we evaluated the integrals

$$\int_0^\infty \frac{\log_e t}{t^2+4}\,dt = \frac{\pi}{4}\log_e 2 \quad \text{and} \quad \int_0^\infty \frac{t^{-1/2}}{t^2+1}\,dt = \frac{\pi}{\sqrt{2}},$$

by using a semicircular contour, indented at the origin. In order for this method to work it was essential that the rational functions in these integrands were both even functions. When faced with an integral such as

$$\int_0^\infty \frac{t^{1/4}}{t^2+t}\,dt,$$

in which the rational function in the integrand is *not* even, some other method must be found.

One possible approach is to make the rational function in the integrand even by using the preliminary substitution $t = s^2 (s = \sqrt{t})$, $dt = 2s\,ds$. In the above example, we obtain

$$\begin{aligned}\int_0^\infty \frac{t^{1/4}}{t^2+t}\,dt &= \int_0^\infty \frac{s^{1/2}}{s^4+s^2}2s\,ds \\ &= 2\int_0^\infty \frac{s^{-1/2}}{s^2+1}\,ds \\ &= 2\left(\frac{\pi}{\sqrt{2}}\right) = \sqrt{2}\pi,\end{aligned}$$

The upper limit ∞ in the second integral is justified by noting that

$$s = \sqrt{t} \to \infty \quad \text{as} \quad t \to \infty.$$

by the result in Frame 2, quoted above.

However, the most widely used method is based on the contour $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$, shown in Figure 1.6. Here Γ_1 and Γ_3 are the same interval $[\varepsilon, r]$ of the positive real axis, traversed in opposite directions, and Γ_2 and Γ_4 are circles with centre 0. Thus the contour Γ is closed, but not simple-closed.

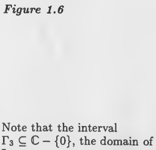


Figure 1.6

We then introduce the function

$$f(z) = \frac{\exp(\frac{1}{4}\operatorname{Log}_{2\pi}(z))}{z^2+z} \qquad (z \in \mathbb{C}_{2\pi}), \tag{1.4}$$

which is analytic on the cut plane $\mathbb{C}_{2\pi}$, and extend the definition of f to Γ_1 and Γ_3 as follows:

$$f(z) = \begin{cases} \dfrac{\exp(\frac{1}{4}\operatorname{Log} z)}{z^2+z} = \dfrac{z^{1/4}}{z^2+z}, & z \in \Gamma_1, \\[2ex] \dfrac{\exp(\frac{1}{4}\operatorname{Log}_{2\pi}(z))}{z^2+z} = \dfrac{z^{1/4}e^{(1/4)2\pi i}}{z^2+z}, & z \in \Gamma_3. \end{cases} \tag{1.5}$$

Note that the interval $\Gamma_3 \subseteq \mathbb{C} - \{0\}$, the domain of $\operatorname{Log}_{2\pi}$.

Strictly speaking, this definition is ambiguous because, as sets, Γ_1 and Γ_3 are equal, and so the extended f is not a function. However, if you follow z round the contour Γ, using the values of f in Equation (1.5) on Γ_1 and Γ_3, then $f(z)$ varies continuously on Γ and behaves like the 'boundary values' of the function f defined by Equation (1.4).

Assuming that the conclusion of the Residue Theorem holds in this situation, we deduce that

$$\begin{aligned}\int_\Gamma f(z)\,dz &= 2\pi i \operatorname{Res}(f,-1)\\ &= (2\pi i \exp(\tfrac{1}{4}\operatorname{Log}_{2\pi}(-1)))/(2(-1)+1) \quad (g/h \text{ Rule})\\ &= -2\pi i e^{(1/4)\pi i},\end{aligned} \tag{1.6}$$

since -1 is the only pole of f 'inside' Γ.

Because Γ is not simple-closed, it does not have an inside. By 'inside' here, we mean that -1 lies in the subset of $\mathbb{C}$ which is on your left as you traverse Γ, as shown in Figure 1.7.

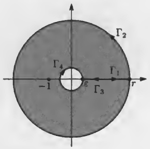


Figure 1.7

As in Frame 2, we can show that

$$\int_{\Gamma_2} f(z)\,dz \to 0 \text{ as } r \to \infty \quad \text{and} \quad \int_{\Gamma_4} f(z)\,dz \to 0 \text{ as } \varepsilon \to 0, \tag{1.7}$$

and Equation (1.5) implies that

$$\int_{\Gamma_1} f(z)\,dz = \int_\varepsilon^r \frac{t^{1/4}}{t^2+t}\,dt \tag{1.8}$$

and

$$\int_{\Gamma_3} f(z)\,dz = e^{(1/2)\pi i}\int_r^\varepsilon \frac{t^{1/4}}{t^2+t}\,dt = -e^{(1/2)\pi i}\int_\varepsilon^r \frac{t^{1/4}}{t^2+t}\,dt. \tag{1.9}$$

Letting $r \to \infty$ and $\varepsilon \to 0$, we deduce from Equations (1.6), (1.7), (1.8) and (1.9) that

$$\left(1 - e^{(1/2)\pi i}\right)\int_0^\infty \frac{t^{1/4}}{t^2+t}\,dt = -2\pi i e^{(1/4)\pi i};$$

that is,

$$\begin{aligned}\int_0^\infty \frac{t^{1/4}}{t^2+t}\,dt &= \frac{-2\pi i e^{(1/4)\pi i}}{1-e^{(1/2)\pi i}}\\ &= \frac{2\pi i}{e^{(1/4)\pi i} - e^{-(1/4)\pi i}}\\ &= \frac{\pi}{\sin\frac{1}{4}\pi} = \sqrt{2}\pi,\end{aligned}$$

as obtained above.

Rather than discuss the justification of this method of evaluation, we simply state a general result which can be obtained by this method. As with the corresponding results in *Unit C1*, Section 3, it is possible to allow simple poles on the positive real axis (by using the Round-the-Pole Lemma).

Theorem 1.1 Let p and q be polynomial functions such that

1. the degree of q exceeds the degree of p by at least 2;
2. any poles of p/q on the non-negative real axis are simple.

Then, for $0 < a < 1$,

$$\int_0^\infty \frac{p(t)}{q(t)} t^a\,dt = -\left(\pi e^{-\pi a i}\operatorname{cosec}\pi a\right)S - (\pi\cot\pi a)T,$$

where S is the sum of the residues of the function

$$f_1(z) = \frac{p(z)}{q(z)}\exp\left(a\operatorname{Log}_{2\pi}(z)\right)$$

in $\mathbb{C}_{2\pi}$, and T is the sum of the residues of the function

$$f_2(z) = \frac{p(z)}{q(z)}\exp\left(a\operatorname{Log} z\right)$$

on the positive real axis.

Note that the non-negative real axis includes 0 but the positive real axis does not.

Problem 1.7

Use Theorem 1.1 to show that

$$\int_0^\infty \frac{t^a}{t^2 - t}\,dt = -\pi \cot \pi a, \qquad \text{for } 0 < a < 1.$$

Remark As well as being used in the evaluation of integrals of the form

$$\int_0^\infty \frac{p(t)}{q(t)} t^a\,dt,$$

the contour in Figure 1.6 can also be used to evaluate integrals of the forms

$$\int_0^\infty \frac{p(t)}{q(t)} \log_e t\,dt \qquad \text{and} \qquad \int_0^\infty \frac{p(t)}{q(t)}\,dt,$$

where p and q are polynomial functions such that the degree of q exceeds that of p by at least 2, p/q is analytic at 0 and the poles of p/q on the positive real axis are simple. We omit the details.

2 INDIRECT ANALYTIC CONTINUATION

After working through this section, you should be able to:

(a) determine *analytic continuations by Taylor series*;

(b) establish that two given functions are indirect analytic continuations of each other;

(c) appreciate how indirect analytic continuation leads to the notions of a *complete analytic function* and a *Riemann surface*.

2.1 Analytic continuation by Taylor series

In the examples given in Subsection 1.1 we were able to find direct analytic continuations by using particular alternative representations of the functions, valid on parts of their domains. In each case, the given alternative representation depended on special knowledge of the function in question (for example, knowing a formula for the sum of a power series). For a general analytic function, however, we need a method that works without recourse to such special knowledge. The following approach has the advantage that it is entirely general, but it has the disadvantage that it is difficult to implement in most particular cases (at least by hand calculations).

Let a function f be analytic on a region $\mathcal{R}$ and consider a point $\alpha \in \mathcal{R}$. Then, by Taylor's Theorem, we know that

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!}(z - \alpha)^n, \qquad \text{for } z \in D, \tag{2.1}$$

where D is any open disc in $\mathcal{R}$ with centre α (see Figure 2.1). But, it may well happen that the Taylor series in Equation (2.1) has a disc of convergence which extends beyond the region $\mathcal{R}$. If this is the case, then the sum function of this power series can be used to provide a direct analytic continuation of f to this disc of convergence. Here is an example of this phenomenon.

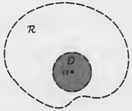


Figure 2.1

Example 2.1

The function $f(z) = \operatorname{Log} z$ is analytic on $\mathcal{R} = \mathbb{C}_\pi$. Determine the Taylor series about $\alpha = -1 + i$ for f and show that the disc of convergence of this Taylor series contains points that are not in $\mathbb{C}_\pi$. Describe the resulting direct analytic continuation g of f.

This Taylor series was discussed briefly in *Unit B3*, Subsection 4.2.

Solution

We have

$$
\begin{aligned}
f(z) &= \operatorname{Log} z, && \text{so } f(-1+i) = \log_e \sqrt{2} + i3\pi/4;\\
f'(z) &= \frac{1}{z}, && \text{so } f'(-1+i) = \frac{1}{-1+i} = \tfrac{1}{2}(-1-i);\\
&\vdots && \qquad \vdots\\
f^{(n)}(z) &= \frac{(-1)^{n-1}(n-1)!}{z^n}, && \text{so } f^{(n)}(-1+i) = \frac{(-1)^{n-1}(n-1)!}{(-1+i)^n}\\
& && \qquad = \frac{(-1)^{n-1}(n-1)!(-1-i)^n}{2^n}\\
& && \qquad = \frac{-(n-1)!(1+i)^n}{2^n}.
\end{aligned}
$$

Hence, the Taylor series about $-1+i$ for f is

$$f(z) = \log_e \sqrt{2} + i3\pi/4 - \sum_{n=1}^{\infty} \frac{(1+i)^n}{n2^n}(z+1-i)^n, \qquad \text{for } z \in D, \qquad (2.2)$$

where D is the largest open disc with centre $-1+i$ in $\mathbb{C}_\pi$, that is, $D = \{z : |z+1-i| < 1\}$; see Figure 2.2.

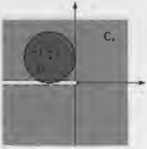


Figure 2.2

We can find the radius of convergence of this power series by using the Ratio Test. Putting

$$a_n = \frac{(1+i)^n}{n2^n}, \qquad n = 1, 2, \ldots,$$

we obtain

$$
\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| &= \lim_{n\to\infty} \left|\frac{(1+i)^n}{n2^n} \cdot \frac{(n+1)2^{n+1}}{(1+i)^{n+1}}\right|\\
&= \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{2}{\sqrt{2}} = \sqrt{2}.
\end{aligned}
$$

Thus, by the Ratio Test, this series has disc of convergence $D' = \{z : |z+1-i| < \sqrt{2}\}$, which contains points which are not in $\mathbb{C}_\pi$ (see Figure 2.3).

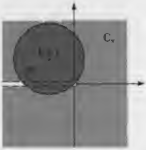


Figure 2.3

The resulting direct analytic continuation of f is therefore

$$g(z) = \log_e \sqrt{2} + i3\pi/4 - \sum_{n=1}^{\infty} \frac{(1+i)^n}{n2^n}(z+1-i)^n \qquad (z \in D'). \quad \blacksquare \quad (2.3)$$

At first sight it may seem surprising that the Taylor series (2.2) has a disc of convergence which extends beyond the region on which the function f is analytic. Notice, however, that f agrees with the function $h(z) = \operatorname{Log}_{2\pi}(z)$ on the open upper half-plane and so f and h must have the same Taylor series about $-1+i$. Since h is analytic on $\mathbb{C}_{2\pi}$, and hence on the open disc $D' = \{z : |z+1-i| < \sqrt{2}\}$, it follows that the Taylor series about $-1+i$ for f must converge to h on D' and so its disc of convergence must include D'. Nor would we expect the disc of convergence to be any larger, since it would then include 0. This would imply that g is bounded near 0, which is not possible because

$$\operatorname{Re} g(z) = \operatorname{Re} h(z) = \log_e |z|, \qquad \text{for } z \in D'.$$

The function g defined in Equation (2.3), above, provides a direct analytic continuation of f from $\mathbb{C}_\pi$ to the open disc $D' = \{z : |z+1-i| < \sqrt{2}\}$. The function g is called a **direct analytic continuation of f by Taylor series.**

The following problem gives you some practice at calculating such direct analytic continuations by Taylor series.

Problem 2.1

Let

$$f(z) = \sum_{n=0}^{\infty} z^n \qquad (|z| < 1)$$

and let $\alpha \in D = \{z : |z| < 1\}$. Determine the Taylor series about α for the function f and hence obtain the corresponding direct analytic continuation of f by Taylor series. Sketch the disc of convergence of this Taylor series for $\alpha = \frac{1}{2}i$.

Example 2.1 illustrates the possible limitations to analytic continuation by Taylor series. The Taylor series in Equation (2.2) has disc of convergence $\{z : |z + 1 - i| < \sqrt{2}\}$, and this disc can be no larger because it cannot enclose the point 0. In a sense, the point 0 acts as a 'barrier' to analytic continuation. The point 1 in Problem 2.1 plays a similar role.

2.2 Complete analytic functions and Riemann surfaces

Having obtained a direct analytic continuation of a given analytic function f with domain $\mathcal{R}$ to a function g with domain $\mathcal{S}$, it is natural to attempt to carry this process further by finding a direct analytic continuation h of g which is not a direct analytic continuation of f (see Figure 2.4).

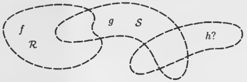


Figure 2.4

For example, if

$$f_1(z) = \mathrm{Log}_{\pi}(z) \qquad (z \in \mathbb{C}_{\pi}),$$
$$f_2(z) = \mathrm{Log}_{2\pi}(z) \qquad (z \in \mathbb{C}_{2\pi}),$$

and

$$f_3(z) = \mathrm{Log}_{3\pi}(z) \qquad (z \in \mathbb{C}_{3\pi}),$$

then

$$f_2(z) = f_1(z), \qquad \text{for } \mathrm{Im}\, z > 0,$$

and

$$f_3(z) = f_2(z), \qquad \text{for } \mathrm{Im}\, z < 0$$

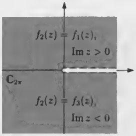


Figure 2.5

(see Figure 2.5). Since the region $\{z : \mathrm{Im} > 0\} \subseteq \mathbb{C}_{\pi} \cap \mathbb{C}_{2\pi}$, f_2 is a direct analytic continuation of f_1 from $\mathbb{C}_{\pi}$ to $\mathbb{C}_{2\pi}$. Similarly, f_3 is a direct analytic continuation of f_2 from $\mathbb{C}_{2\pi}$ to $\mathbb{C}_{3\pi}$. But

$$f_3(z) = f_1(z) + 2\pi i, \qquad \text{for } z \in \mathbb{C}_{\pi} = \mathbb{C}_{3\pi},$$

so f_3 is *not* a direct analytic continuation of f_1. However, there is clearly a sense in which f_3 is an 'analytic continuation' of f_1 and so we extend our definition to include this phenomenon. To emphasize the importance of

domains in this definition, we shall use the notation $(f, \mathcal{R})$ to denote an analytic function f whose domain is the region $\mathcal{R}$.

The pair $(f, \mathcal{R})$ is sometimes called a 'function element'.

> **Definitions** The finite sequence of functions
>
> $$(f_1, \mathcal{R}_1), (f_2, \mathcal{R}_2), \ldots, (f_n, \mathcal{R}_n)$$
>
> forms a **chain** if, for $k = 1, 2, \ldots, n-1$,
>
> $$(f_{k+1}, \mathcal{R}_{k+1}) \text{ is a direct analytic continuation of } (f_k, \mathcal{R}_k).$$
>
> Any two functions of the chain are called **analytic continuations of each other**, and the chain is said to **join** $(f_1, \mathcal{R}_1)$ to $(f_n, \mathcal{R}_n)$. If $\mathcal{R}_1 = \mathcal{R}_n$, then the chain is said to be **closed**.

Note that f_1 need not equal f_n for a chain to be complete.

These definitions are illustrated in Figures 2.6 and 2.7.

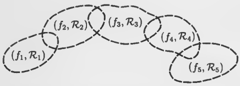


Figure 2.6 A chain of functions which joins $(f_1, \mathcal{R}_1)$ to $(f_5, \mathcal{R}_5)$

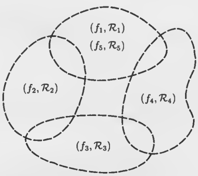


Figure 2.7 A closed chain, with $\mathcal{R}_1 = \mathcal{R}_5$

Two functions of a chain which are not direct analytic continuations of each other are called **indirect analytic continuations**. For example, the functions

$$f_1(z) = \mathrm{Log}_\pi(z) \quad (z \in \mathbb{C}_\pi) \qquad \text{and} \qquad f_3(z) = \mathrm{Log}_{3\pi}(z) \quad (z \in \mathbb{C}_{3\pi}),$$

discussed earlier, are indirect analytic continuations of each other and the sequence $(f_1, \mathbb{C}_\pi), (f_2, \mathbb{C}_{2\pi}), (f_3, \mathbb{C}_{3\pi})$, where

$$f_k(z) = \mathrm{Log}_{k\pi}(z), \qquad \text{for } k = 1, 2, 3,$$

forms a closed chain because $\mathbb{C}_{3\pi} = \mathbb{C}_\pi$ (even though $f_1 \neq f_3$ on this set).

Problem 2.2

Consider the functions

$$f_k(z) = \exp\left(\tfrac{1}{2}\mathrm{Log}_{k\pi}(z)\right) \qquad (z \in \mathbb{C}_{k\pi}), \text{ for } k \in \mathbb{Z}.$$

Note that

$$f_1(z) = \sqrt{z}, \quad \text{for } z \in \mathbb{C}_\pi.$$

(a) Show that $(f_1, \mathbb{C}_\pi), (f_2, \mathbb{C}_{2\pi}), (f_3, \mathbb{C}_{3\pi})$ form a closed chain, but $f_1 \neq f_3$.

(b) Show that $(f_1, \mathbb{C}_\pi), (f_2, \mathbb{C}_{2\pi}), (f_3, \mathbb{C}_{3\pi}), (f_4, \mathbb{C}_{4\pi}), (f_5, \mathbb{C}_{5\pi})$ form a closed chain, and $f_1 = f_5$.

Problem 2.3

Let $D_0 = \{z : |z| < 1\}$ and $D_2 = \{z : |z - 2| < 1\}$. Show that the functions

$$f(z) = \sum_{n=0}^{\infty} z^n \quad (z \in D_0) \qquad \text{and} \qquad g(z) = \sum_{n=0}^{\infty} (-1)^{n+1}(z-2)^n \quad (z \in D_2)$$

are indirect analytic continuations of each other.
(*Hint*: Use the result of Example 1.1.)

Suppose now that a function $(f, \mathcal{R})$ is given. Then, as we have seen, different analytic continuations of $(f, \mathcal{R})$ to the same region may lead to different functions (see Problem 2.2(a) with $(f, \mathcal{R}) = (f_2, \mathbb{C}_{2\pi})$, for example). This leads us to introduce the notion of a **complete analytic function** corresponding to the function $(f, \mathcal{R})$, which is the set of all functions which are analytic continuations of $(f, \mathcal{R})$, both direct and indirect. This name, however, is rather misleading because, as defined, a complete analytic function is not actually a function, but a *set* of functions, which are related to each other via analytic continuation.

In the rest of this section, which is intended for reading only, you should aim for a general understanding rather than mastery of all the details.

In order to interpret a complete analytic function as a single function, Riemann introduced the idea of constructing a surface to act as the domain of a complete analytic function. We describe one way to carry out this construction in the case of the function $(f_1, \mathbb{C}_\pi)$ where

Georg Friedrich Bernhard Riemann (1826–1866), who studied at Göttingen with Gauss and later with Dirichlet and Jacobi in Berlin, was one of the greatest mathematicians of all time. Throughout his life he had poor health, and died at the age of 39.

$$f_1(z) = \mathrm{Log}_\pi(z).$$

Consider the functions $(f_k, \mathbb{C}_{k\pi})$, $k \in \mathbb{Z}$, where

$$f_k(z) = \mathrm{Log}_{k\pi}(z),$$

each of which is an analytic continuation of $(f_1, \mathbb{C}_\pi)$, because

$$f_{k+1}(z) = f_k(z), \qquad \text{for } \begin{cases} \mathrm{Im}\, z > 0, & \text{if } k \text{ is odd,} \\ \mathrm{Im}\, z < 0, & \text{if } k \text{ is even.} \end{cases}$$

The cut planes $\ldots, \mathbb{C}_{-\pi}, \mathbb{C}_\pi, \mathbb{C}_{3\pi}, \ldots$ are all the same set, although we have shown them as separate copies in Figure 2.8 and we shall think of them as separate in what follows.

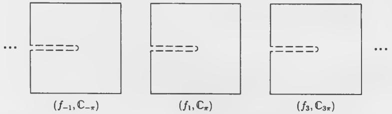


Figure 2.8

Our earlier discussion shows that $(f_1, \mathbb{C}_\pi)$ and $(f_3, \mathbb{C}_{3\pi})$ are both direct analytic continuations of $(f_2, \mathbb{C}_{2\pi})$. Moreover, there is a sense in which $(f_3, \mathbb{C}_{3\pi})$ is obtained from $(f_1, \mathbb{C}_\pi)$ by analytic continuation of f_1 across the negative real axis; indeed, the values of f_1 just above the cut (which are equal to the values of f_2 there) 'match' the values of f_3 just below the cut (which are equal to the values of f_2 there). (See Figure 2.9.)

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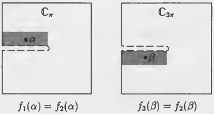


Figure 2.9

This suggests that we should 'sew' the upper edge of the cut in $\mathbb{C}_\pi$ to the lower edge of the cut in $\mathbb{C}_{3\pi}$, producing a 'seam' along the negative real axis. We can then define a continuous function by using

Note that the 'seam' does not contain 0.

the values of f_1 on $\mathbb{C}_\pi$,

the values of f_3 on $\mathbb{C}_{3\pi}$,

the values of f_2 on the 'seam'.

We now apply a similar process to all the adjacent pairs of cut planes in Figure 2.8. We can picture the resulting surface as a set of infinitely many vertically stacked sheets (one for each cut plane $\mathbb{C}_{(2k-1)\pi}, k \in \mathbb{Z}$) sewn together as described above. This surface is rather like a flattened multi-storey car park, as shown in Figure 2.10, extending indefinitely far upwards, downwards and outwards.

You might like to try to make such a surface with sheets of paper and adhesive tape.

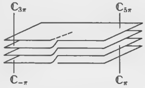


Figure 2.10

We can now define a continuous function, f say, on the whole of this surface, as follows:

$$f = \begin{cases} f_{2k-1} & \text{on the part of the surface containing } \mathbb{C}_{(2k-1)\pi}, \\ f_{2k} & \text{on the 'seam' joining } \mathbb{C}_{(2k-1)\pi} \text{ to } \mathbb{C}_{(2k+1)\pi}, \end{cases}$$

for each $k \in \mathbb{Z}$.

It can be shown that

> any function in a chain starting from $(f_1, \mathbb{C}_\pi)$ can be represented on the surface and will agree with f there.

For example, $(f_2, \mathbb{C}_{2\pi})$ can be represented on the shaded part of the surface shown in Figure 2.11.

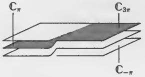


Figure 2.11

Thus the surface can be thought of as the domain of the complete analytic function corresponding to $(f_1, \mathbb{C}_\pi)$. With this interpretation, we speak of this surface as a **Riemann surface** for the complete analytic function f and we think of f as a representation of the 'multi-valued logarithm function'.

The sewn-sheets representation of a Riemann surface is not the only one. For example, the Riemann surface for the complete analytic function corresponding to the function $(\mathrm{Log}_\pi, \mathbb{C}_\pi)$ is shown in some texts as an infinite spiral surface. Figure 2.2(a) of *Unit A2* shows part of such a surface.

The theory of Riemann surfaces can be taken very much further and placed on a sound theoretical basis (as opposed to the descriptive approach given here). It is then possible to transfer much of the theory of complex analysis from the complex plane $\mathbb{C}$ to a general Riemann surface and hence gain much useful information about those functions (usually inverse functions, such as $z \longmapsto \mathrm{Log}\, z$ and $z \longmapsto \sqrt{z}$) whose analytic continuations cannot be represented by an ordinary analytic function whose domain is a region in the complex plane.

3 UNIFORM CONVERGENCE

After working through this section, you should be able to:

(a) understand the definition of *uniform convergence*;

(b) use Weierstrass' M-test to prove that various series of functions are uniformly convergent;

(c) use Weierstrass' Theorem to prove that various functions defined by series are analytic and to find their derivatives;

(d) define the *zeta function* ζ and understand the role of Weierstrass' Theorem in proving that ζ is analytic.

3.1 The zeta function

Representation by Taylor series and, more generally, by Laurent series is a very useful technique for exploiting analytic functions. For example, it is the basis of the Residue Theorem which, as you have seen, has many applications. The disadvantage of such series, however, is that they can be used to represent analytic functions only on an open disc or an open annulus and, for some important analytic functions, such sets are inappropriate. For example, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^z} = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots, \tag{3.1}$$

Recall that $\frac{1}{n^z} = n^{-z} = e^{-z \log_e n}$.

considered in *Unit B3*, Example 1.5, is absolutely convergent for $\operatorname{Re} z > 1$. Thus the sum function for the series in Equation (3.1) can be used to define a function with domain the open half-plane $\{z : \operatorname{Re} z > 1\}$. This rather strange function is of great importance in number theory, a fact first recognized (for real z) by Euler. For this reason it was extensively studied in the 19th century by Riemann, who called it the **zeta function**:

$$\zeta(z) = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots \qquad (\operatorname{Re} z > 1). \tag{3.2}$$

We have already calculated several values of the zeta function using the residue calculus. For example,

Unit C1, Example 4.1, and Problem 4.1

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{and} \quad \zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

The same technique can be used to determine $\zeta(6), \zeta(8), \ldots$, but most values of the function ζ can be found only approximately.

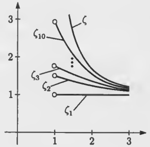


Figure 3.1

The definition of the zeta function in Equation (3.2) may be expressed as

$$\zeta(z) = \lim_{n \to \infty} \zeta_n(z) \qquad (\operatorname{Re} z > 1),$$

where $\{\zeta_n\}$ is the sequence of partial sum functions

$$\zeta_n(z) = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \cdots + \frac{1}{n^z}, \qquad (z \in \mathbb{C}), n = 1, 2, \ldots. \tag{3.3}$$

The graphs of the restrictions of some of these partial sum functions to $\{x \in \mathbb{R} : x > 1\}$ are shown in Figure 3.1. The graph of the 'limit function' $x \longmapsto \zeta(x)$, for x near 1, shows why the restriction $\operatorname{Re} z > 1$ is needed in Equation (3.2).

Since each of the functions ζ_n is analytic on $\mathbb{C}$ (because $1/n^z = \exp(-z \log_e n)$), it seems likely that

the zeta function is analytic on $\{z : \operatorname{Re} z > 1\}$.

However, the only result that we have proved so far about a function defined by a series being analytic is the Differentiation Rule for power series. Since the series in Equation (3.2) is not a power series, this theorem cannot be applied.

Unit B3, Theorem 2.2

In the next subsection we develop a type of convergence which will enable us to prove that the zeta function is indeed analytic.

3.2 Convergence of sequences of functions

There are various ways in which a sequence of functions $\{f_n\}$ may converge to a limit function f; the following one is perhaps the simplest.

> **Definition** A sequence of functions $\{f_n\}$ **converges pointwise** (to a **limit function f**) **on a set E** if, for each $z \in E$,
>
> $$\lim_{n\to\infty} f_n(z) = f(z). \tag{3.4}$$

For example, the sequence of functions ζ_n defined by Equation (3.3) converges pointwise to the zeta function ζ on the set $E = \{z : \operatorname{Re} z > 1\}$. A simpler example is the sequence of functions

$$f_n(z) = z^n, \qquad n = 1, 2, \ldots .$$

Since $z^n \to 0$ as $n \to \infty$, for $|z| < 1$, we deduce that the sequence $\{f_n\}$ converges pointwise to the zero function $f(z) = 0$ on the set $\{z : |z| < 1\}$. This behaviour is illustrated for real values of x in the interval $[0, 1]$ in Figure 3.2. Notice that $f_n(x) \to 0$ as $n \to \infty$, for $0 \le x < 1$, whereas $f_n(1) \to 1$ as $n \to \infty$.

Unit A3, Theorem 1.2(b)

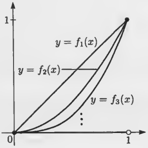


Figure 3.2

Though useful, pointwise convergence is unfortunately not strong enough to give the type of convergence theorem that we require. Instead, we introduce a type of convergence which guarantees that $f_n(z) \to f(z)$ as $n \to \infty$ at a *uniform rate* as z varies over a set.

> **Definition** A sequence of functions $\{f_n\}$ **converges uniformly** (to a **limit function f**) **on a set E** if
>
> for each $\varepsilon > 0$, there is an integer N such that
>
> $$|f_n(z) - f(z)| < \varepsilon, \qquad \text{for all } n > N, \text{ and all } z \in E. \tag{3.5}$$
>
> We also say that $\{f_n\}$ is **uniformly convergent on E, with limit function f**.

This definition is illustrated in Figure 3.3 for two arbitrary values z_1 and z_2 of z in E, a given $\varepsilon > 0$ and $n = N + 1$. The sequences

$$f_n(z_1),\ n = N+1, N+2, \ldots, \quad \text{and} \quad f_n(z_2),\ n = N+1, N+2, \ldots,$$

lie in the discs of radius ε with centres $f(z_1)$ and $f(z_2)$, repectively, and converge to those values.

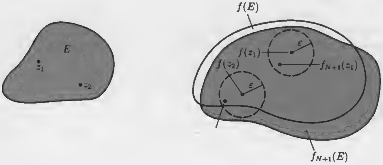


Figure 3.3

Remarks

1 To appreciate how the definition of uniform convergence differs from that of pointwise convergence, it helps to recast Condition (3.4) in the equivalent form:

for each $\varepsilon > 0$, there is an integer N such that

$$|f_n(z) - f(z)| < \varepsilon, \qquad \text{for all } n > N.$$

The choice of N here depends on the given ε *and* on z; whereas in the definition of uniform convergence the choice of N depends only on the given ε — the same N works for all $z \in E$. Therein lies the *uniformity* of the convergence.

2 It is clear that if $\{f_n\}$ converges uniformly on E, then it converges uniformly on any subset of E. Also, if $\{f_n\}$ converges uniformly to f on E, then $\{f_n\}$ converges pointwise to f on E.

Example 3.1

Prove that the sequence $f_n(z) = z^n, n = 1, 2, \ldots$, converges uniformly on $E = \{z : |z| \le \frac{1}{2}\}$.

Solution

For each $z \in E$, we have $|z| \le \frac{1}{2}$ and so

$$f_n(z) = z^n \to 0 \text{ as } n \to \infty.$$

Hence $\{f_n\}$ converges pointwise to the function $f(z) = 0$ on E.

Now

$$|f_n(z) - f(z)| = |z|^n \le (\tfrac{1}{2})^n, \qquad \text{for } n = 1, 2, \ldots, \text{ and all } z \in E.$$

Thus to satisfy Inequality (3.5) we need only choose N so large that

$$(\tfrac{1}{2})^n < \varepsilon, \qquad \text{for all } n > N,$$

and since $\{(\frac{1}{2})^n\}$ is a (basic) null sequence, this is clearly possible.

Hence $\{f_n\}$ converges uniformly to f on E. ■

The solution to Example 3.1 illustrates one way to prove that a sequence of functions converges uniformly, which we summarize in the following strategy.

> **Strategy for proving uniform convergence**
>
> To prove that a sequence of functions $\{f_n\}$ converges uniformly on a set E
>
> (a) determine the limit function f by evaluating
>
> $$f(z) = \lim_{n\to\infty} f_n(z), \qquad \text{for } z \in E;$$
>
> (b) find a null sequence $\{a_n\}$ of positive terms such that
>
> $$|f_n(z) - f(z)| \le a_n, \qquad \text{for } n = 1, 2, \ldots, \text{ and all } z \in E.$$

Using this strategy we can show, as in the above solution, that the sequence of functions $f_n(z) = z^n, n = 1, 2, \ldots$, converges uniformly to the function $f(z) = 0$ on *any* closed disc of the form $E = \{z : |z| \le r\}$, where $0 < r < 1$. In this case, the null sequence in step (b) of the strategy is $\{r^n\}$. However, this sequence of functions does not converge uniformly to its limit function $f(z) = 0$ on the set $E = \{z : |z| < 1\}$. Roughly speaking, this is because $z^n \to 0$ as $n \to \infty$ more and more slowly as z approaches the boundary of $\{z : |z| < 1\}$ (see Figure 3.2). More precisely, if $\varepsilon = \frac{1}{2}$ say, then there is no positive integer N such that

$$|f_n(z) - f(z)| = |z|^n < \tfrac{1}{2}, \qquad \text{for all } n > N, \text{ and all } z \in E,$$

because, for any given positive integer n,

$$|z| \ge (\tfrac{1}{2})^{1/n} \implies |z|^n \ge \tfrac{1}{2}.$$

Problem 3.1

Prove that the sequence

$$f_n(z) = \frac{1}{1+z^n}, \qquad n = 1, 2, \ldots,$$

converges uniformly on $E = \{z : |z| \leq r\}$, for $0 < r < 1$.

Recalling our aim of proving that the zeta function is analytic, we now adapt the concepts of pointwise and uniform convergence to series of functions. The various conventions associated with series of complex numbers are readily adapted to series of complex functions. For instance, if necessary, such a series may start with a value of n other than 1.

Definition If $\{\phi_n\}$ is a sequence of functions, then the series of functions

$$\sum_{n=1}^{\infty} \phi_n = \phi_1 + \phi_2 + \cdots$$

converges pointwise/uniformly on a set E if the sequence of **partial sum functions**

$$f_n(z) = \phi_1(z) + \phi_2(z) + \cdots + \phi_n(z)$$

converges pointwise/uniformly on E, respectively. The limit function f of the sequence $\{f_n\}$ is called the **sum function** of $\sum_{n=1}^{\infty} \phi_n$ **on E**, written

$$f(z) = \sum_{n=1}^{\infty} \phi_n(z) \qquad (z \in E).$$

This definition agrees with that of the sum function of a power series on a set, given in *Unit B3*, Section 2.

For example, if

$$\phi_n(z) = z^n, \qquad n = 0, 1, 2, \ldots,$$

then the partial sum functions of $\sum_{n=0}^{\infty} \phi_n$ are

$$\begin{aligned} f_n(z) &= \phi_0(z) + \phi_1(z) + \cdots + \phi_n(z) \\ &= 1 + z + \cdots + z^n \\ &= \frac{1 - z^{n+1}}{1 - z}, \qquad \text{for } z \neq 1. \end{aligned}$$

Hence the sum function of $\sum_{n=0}^{\infty} \phi_n$ on $\{z : |z| < 1\}$ is

$$f(z) = \lim_{n\to\infty} f_n(z) = \frac{1}{1-z} \qquad (|z| < 1).$$

In view of Example 3.1 and the discussion following it, we might expect the convergence of this series to be uniform on each set $\{z : |z| \leq r\}$, where $0 < r < 1$. Rather than prove this directly, we can use the following test for uniform convergence of series, which is widely applicable.

> **Theorem 3.1 Weierstrass' *M*-test**
>
> Let $\{\phi_n\}$ be a sequence of functions defined on a set E and suppose that there is a sequence of positive terms $\{M_n\}$, such that
>
> 1. $|\phi_n(z)| \le M_n$, for $n = 1, 2, \ldots$, and all $z \in E$;
> 2. $\sum_{n=1}^{\infty} M_n$ is convergent.
>
> Then the series $\sum_{n=1}^{\infty} \phi_n$ is uniformly convergent on E.

We shall refer to this result as the 'M-test', since we introduce Weierstrass' Theorem in the next subsection.

Proof First note that, by Assumptions 1 and 2, $\sum_{n=1}^{\infty} |\phi_n(z)|$ is convergent for each $z \in E$, by the Comparison Test. Hence $\sum_{n=1}^{\infty} \phi_n(z)$ is convergent for each $z \in E$, by the Absolute Convergence Test, and so the sum function

Unit B3, Theorem 1.6

Unit B3, Theorem 1.7

$$f(z) = \sum_{n=1}^{\infty} \phi_n(z)$$

exists for each $z \in E$. As usual, we denote the partial sum functions by

$$f_n(z) = \phi_1(z) + \phi_2(z) + \cdots + \phi_n(z), \qquad n = 1, 2, \ldots.$$

Then, for $z \in E$,

$$\begin{aligned} |f(z) - f_n(z)| &= |\phi_{n+1}(z) + \phi_{n+2}(z) + \cdots| \\ &= \left| \sum_{k=n+1}^{\infty} \phi_k(z) \right| \\ &\le \sum_{k=n+1}^{\infty} |\phi_k(z)| \qquad \text{(by the Triangle Inequality)} \\ &\le \sum_{k=n+1}^{\infty} M_k \qquad \text{(by Assumption 1).} \end{aligned}$$

Unit B3, Theorem 1.8

By Assumption 2, however,

$$\sum_{k=n+1}^{\infty} M_k \to 0 \text{ as } n \to \infty,$$

(because $\sum_{k=n+1}^{\infty} M_k = \sum_{k=1}^{\infty} M_k - \sum_{k=1}^{n} M_k \to 0$ as $n \to \infty$), so the uniform convergence of $\sum_{n=1}^{\infty} \phi_n$ follows from the strategy given on page 22. ■

Example 3.2

Prove that the series $\sum_{n=0}^{\infty} z^n$ converges uniformly on $E = \{z : |z| \le r\}$, for $0 < r < 1$.

Solution

Here

$$\phi_n(z) = z^n, \qquad n = 0, 1, 2, \ldots,$$

and

$$|\phi_n(z)| = |z|^n \le r^n, \qquad \text{for } z \in E.$$

Hence Assumption 1 of the M-test holds with $M_n = r^n, n = 0, 1, 2, \ldots$. Since

$$\sum_{n=0}^{\infty} M_n = \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \qquad \text{for } 0 < r < 1,$$

Assumption 2 of the M-test also holds, and so $\sum_{n=0}^{\infty} z^n$ is uniformly convergent on E. ■

Example 3.2 shows that the power series $\sum_{n=0}^{\infty} z^n$ converges uniformly on each closed disc lying inside its disc of convergence. In fact this result holds for an arbitrary power series, as we now ask you to verify (for simplicity the power series is about $\alpha = 0$).

Problem 3.2

Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with disc of convergence $\{z : |z| < R\}$, where $R > 0$. Show that the power series is uniformly convergent on each closed disc $\{z : |z| \le r\}$, where $0 < r < R$.

(*Hint*: Use the fact that the power series converges absolutely at each point in its disc of convergence.)

Unit B3, Theorem 2.1

The next example shows that the series for the zeta function is uniformly convergent on each closed half-plane lying in $\{z : \operatorname{Re} z > 1\}$. (We shall use this result in the next subsection when we prove that the zeta function is analytic on $\{z : \operatorname{Re} z > 1\}$.)

Example 3.3

Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^z}$ converges uniformly on $E = \{z : \operatorname{Re} z \ge 1 + \varepsilon\}$, for $\varepsilon > 0$ (see Figure 3.4).

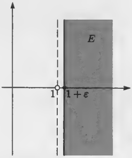


Figure 3.4

Solution

Here

$$\phi_n(z) = \frac{1}{n^z} = e^{-z \log_e n}, \qquad n = 1, 2, \ldots,$$

and

$$|\phi_n(z)| = e^{-(\operatorname{Re} z) \log_e n} = \frac{1}{n^{\operatorname{Re} z}} \le \frac{1}{n^{1+\varepsilon}}, \qquad \text{for } z \in E.$$

Hence Assumption 1 of the M-test holds with $M_n = 1/n^{1+\varepsilon}, n = 1, 2, \ldots$. Since

$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}}$$

is convergent, for $\varepsilon > 0$, Assumption 2 of the M-test also holds, and so $\sum_{n=1}^{\infty} \frac{1}{n^z}$ is uniformly convergent on E. ■

Unit B3, Theorem 1.3

Problem 3.3

Prove that the series

$$\sum_{n=1}^{\infty} \frac{z^n}{n^2}$$

is uniformly convergent on $E = \{z : |z| \le 1\}$.

3.3 Weierstrass' Theorem

We now give a very powerful result about uniformly convergent sequences of analytic functions. In addition to proving that the zeta function is analytic, this result will enable us to introduce another method of representing analytic functions, which will be discussed in Section 4.

> **Theorem 3.2 Weierstrass' Theorem**
>
> Let $\{f_n\}$ be a sequence of functions which are analytic on a region $\mathcal{R}$ and which converge uniformly to a function f on each closed disc in $\mathcal{R}$. Then
>
> f is analytic on $\mathcal{R}$
>
> and
>
> the sequence $\{f'_n\}$ converges uniformly to f' on each closed disc in $\mathcal{R}$.

Remarks

1 The hypothesis that $\{f_n\}$ converges uniformly on 'each closed disc in $\mathcal{R}$' may seem strange. However, this often occurs in practice even though $\{f_n\}$ may not converge uniformly to f on the whole of $\mathcal{R}$ (see Example 3.4).

2 Notice that if $\{f'_n\}$ converges uniformly to f' on each closed disc in $\mathcal{R}$, then $\{f'_n\}$ converges pointwise to f' on the whole of $\mathcal{R}$.

3 Having applied Weierstrass' Theorem to a suitable sequence $\{f_n\}$, we can then apply it to the sequence $\{f'_n\}$ and hence deduce that $\{f''_n\}$ converges uniformly to f'' on each closed disc in $\mathcal{R}$, and so on.

The proof of this result uses many earlier results and techniques, such as contour integration, Cauchy's Theorem, Morera's Theorem, Cauchy's First Derivative Formula, and the ubiquitous Estimation Theorem. Before giving the proof, we use Theorem 3.2 to prove that the zeta function is analytic.

Example 3.4

Prove that the zeta function ζ is analytic on $\{z : \operatorname{Re} z > 1\}$ and obtain a formula for ζ'.

Solution

In Example 3.3 we saw that the series

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

is uniformly convergent on any closed half-plane $\{z : \operatorname{Re} z \geq 1 + \varepsilon\}$, for $\varepsilon > 0$, and so this series is uniformly convergent on each closed disc in $\{z : \operatorname{Re} z > 1\}$. See Remark 2, page 22.
Since the partial sum functions

$$\begin{aligned}\zeta_n(z) &= 1 + \frac{1}{2^z} + \cdots + \frac{1}{n^z}\\ &= 1 + e^{-z\log_e 2} + \cdots + e^{-z\log_e n}, \qquad n = 1, 2, \ldots,\end{aligned}$$

are each analytic on $\mathbb{C}$, it follows from Weierstrass' Theorem that ζ is analytic on $\{z : \operatorname{Re} z > 1\}$.

To obtain a formula for ζ' we note from Remark 2 above that ζ'_n converges pointwise to ζ' on $\{z : \operatorname{Re} z > 1\}$. Now

$$\begin{aligned}\zeta'_n(z) &= 0 - (\log_e 2)e^{-z\log_e 2} - \cdots - (\log_e n)e^{-z\log_e n}\\ &= -\left(\frac{\log_e 2}{2^z} + \cdots + \frac{\log_e n}{n^z}\right),\end{aligned}$$

and so

$$\zeta'(z) = -\sum_{n=2}^{\infty} \frac{\log_e n}{n^z}, \qquad \text{for } \operatorname{Re} z > 1. \quad \blacksquare$$

Remarks

1 As indicated by this example, when Weierstrass' Theorem is used to prove that a function f defined by a series is analytic, the derivative f' may be obtained from term by term differentiation of the series.

2 In Section 5 you will see that the zeta function can be analytically continued to the region $\mathbb{C} - \{1\}$ and that it has a simple pole at the point 1.

Problem 3.4

Use the result of Problem 3.2 to prove that if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has disc of convergence $\{z : |z| < R\}$, then

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}, \qquad \text{for } |z| < R.$$

This is an alternative proof of part of the Differentiation Rule (*Unit B3*, Theorem 2.2).

Problem 3.5

Obtain a formula for $\zeta''(z)$.

Proof of Weierstrass' Theorem

This proof may be omitted on a first reading.

We shall prove Weierstrass' Theorem under the stronger assumption that the sequence $\{f_n\}$ converges uniformly to f on the whole of $\mathcal{R}$. The (more general) version stated in Theorem 3.2 can then be deduced by applying this special case to open discs in $\mathcal{R}$. We omit the details.

To prove that the limit function f is analytic we want to apply Morera's Theorem. This means that we shall need to integrate along contours in $\mathcal{R}$, and so we must begin by proving that f is continuous on $\mathcal{R}$.

Unit B2, Theorem 5.4

The proof is in four steps.

(a) To prove that f is continuous at each point $\alpha \in \mathcal{R}$, we need to show that for each $\varepsilon > 0$ there is $\delta > 0$ such that

$$z \in \mathcal{R}, |z - \alpha| < \delta \implies |f(z) - f(\alpha)| < \varepsilon.$$

By the uniform convergence of $\{f_n\}$ to f, we can choose N so that

$$|f_n(z) - f(z)| < \tfrac{1}{3}\varepsilon, \qquad \text{for all } n > N, \text{ and all } z \in \mathcal{R}.$$

In particular,

$$|f_n(\alpha) - f(\alpha)| < \tfrac{1}{3}\varepsilon, \qquad \text{for all } n > N.$$

Now choose some fixed $n > N$. Then, since the function f_n is continuous at α, we can choose $\delta > 0$ so that

$$z \in \mathcal{R}, |z - \alpha| < \delta \implies |f_n(z) - f_n(\alpha)| < \tfrac{1}{3}\varepsilon.$$

It follows, by the Triangle Inequality, that, for $z \in \mathcal{R}$ and $|z - \alpha| < \delta$,

$$\begin{aligned} |f(z) - f(\alpha)| &= |(f(z) - f_n(z)) + (f_n(z) - f_n(\alpha)) + (f_n(\alpha) - f(\alpha))| \\ &\le |f(z) - f_n(z)| + |f_n(z) - f_n(\alpha)| + |f_n(\alpha) - f(\alpha)| \\ &< \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon = \varepsilon, \end{aligned}$$

as required. Hence f is continuous at α.

(b) Next we show that if Γ is a contour in $\mathcal{R}$, then

$$\lim_{n\to\infty} \int_\Gamma f_n(z)\,dz = \int_\Gamma f(z)\,dz. \tag{3.6}$$

(Note that the integral on the right *is* defined, by step (a).) By the uniform convergence of $\{f_n\}$ to f, for each $\varepsilon > 0$, there is an integer N such that

$$|f_n(z) - f(z)| < \varepsilon, \qquad \text{for all } n > N, \text{ and all } z \in \mathcal{R}.$$

Thus, if $n > N$ and L is the length of Γ, then

$$\left|\int_\Gamma f_n(z)\,dz - \int_\Gamma f(z)\,dz\right| = \left|\int_\Gamma (f_n(z) - f(z))\,dz\right| \le \varepsilon L,$$

by the Estimation Theorem. Since this holds for any $\varepsilon > 0$, Equation (3.6) follows.

(c) Now we show that f is analytic on each open disc D in $\mathcal{R}$ and hence throughout $\mathcal{R}$. Let Γ be any rectangular contour in D. Then, by step (b) and Cauchy's Theorem,

Unit B2, Theorem 1.2

$$\int_\Gamma f(z)\,dz = \lim_{n\to\infty} \int_\Gamma f_n(z)\,dz = 0,$$

since each of the functions f_n is analytic on D. Hence, by Morera's Theorem, f itself is analytic on D, as required.

(d) Finally, we show that $\{f_n'\}$ converges uniformly to f' on each closed disc in $\mathcal{R}$. Let $E = \{z : |z - \alpha| \le r\}$ be a closed disc in $\mathcal{R}$, and choose $\rho > r$ so that the circle C with centre α and radius ρ and its inside lie in $\mathcal{R}$ (see Figure 3.5).

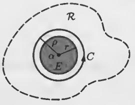


Figure 3.5

Then, for $z \in E$, we can use Cauchy's First Derivative Formula to write

$$f_n'(z) - f'(z) = \frac{1}{2\pi i}\int_C \frac{f_n(w) - f(w)}{(w-z)^2}\,dw.$$

Now $\{f_n\}$ converges uniformly to f on $\mathcal{R}$, and hence also on C. Thus, given $\varepsilon > 0$, we can choose an integer N so that

Since w is the integration variable here, we think of C as $\{w : |w - \alpha| = \rho\}$.

$$|f_n(w) - f(w)| < \varepsilon, \qquad \text{for all } n > N, \text{ and all } w \in C.$$

Also, for $w \in C$,

$$|w - z| = |(w - \alpha) + (\alpha - z)| \ge |w - \alpha| - |z - \alpha| = \rho - r,$$

by the Triangle Inequality. Hence, by the Estimation Theorem,

$$\begin{aligned} |f_n'(z) - f_n(z)| &= \left|\frac{1}{2\pi i}\int_C \frac{f_n(w) - f(w)}{(w-z)^2}\,dw\right| \\ &\le \frac{1}{2\pi}\cdot\frac{\varepsilon}{(\rho - r)^2}\cdot 2\pi\rho = \frac{\rho\varepsilon}{(\rho - r)^2}, \qquad \text{for all } n > N. \end{aligned}$$

Since ρ and r are fixed numbers, independent of the choice of $z \in E$, we deduce that $\{f_n'\}$ converges uniformly to f on E. ∎

4 THE GAMMA FUNCTION

After working through this section, you should be able to:

(a) define the *gamma function* as an improper integral;

(b) differentiate under an integral sign, under appropriate conditions;

(c) use properties of the gamma function, such as its functional equation;

(d) understand how the gamma function is analytically continued.

In *Unit A2* we introduced the so-called 'elementary functions' of complex analysis: polynomial functions, rational functions, the exponential function, trigonometric functions and hyperbolic functions, and some of their inverse

functions. In this section, we introduce the next most important function in complex analysis, the gamma function, which often arises in the evaluation of complex integrals.

4.1 Defining the gamma function

The gamma function provides an answer to the following simple question.

Is it possible to define $z!$ for complex z?

More precisely:

Is there a function f, with domain a region $\mathcal{R}$ containing the set $\{0, 1, 2, \ldots\}$ such that

(a) f is analytic on $\mathcal{R}$,

(b) $f(n) = n!$, for $n = 0, 1, 2, \ldots$,

(c) $f(z+1) = (z+1)f(z)$, for $z \in \mathcal{R}$?

By convention, $0! = 1$.

The points $(n, n!)$ are plotted on the (x, y)-plane in Figure 4.1, together with a smooth curve through them which suggests that such a function f may exist — at least for positive real values x. It also suggests that such a function f should satisfy

$$0 < f(x) < 1, \qquad \text{for } 0 < x < 1.$$

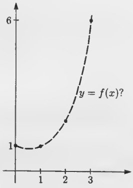


Figure 4.1

In fact, there are many ways to define such a function f, but the most useful one is the *gamma function*. Euler defined this function by using certain improper integrals and we shall adopt a similar approach.

Consider the sequence of improper integrals

$$I_n = \int_0^\infty e^{-t} t^n \, dt, \qquad n = 0, 1, 2, \ldots. \tag{4.1}$$

You have already seen that

$$I_0 = \lim_{r \to \infty} \int_0^r e^{-t} dt = 1,$$

Unit C1, Problem 3.3(c)

and we now evaluate I_n, for $n = 1, 2, \ldots$, by obtaining a reduction formula. For $r > 0$, we have, by integration by parts,

$$\begin{aligned} \int_0^r e^{-t} t^n \, dt &= \left[-e^{-t} t^n\right]_0^r + n \int_0^r e^{-t} t^{n-1} \, dt \\ &= -e^{-r} r^n + n \int_0^r e^{-t} t^{n-1} \, dt, \qquad \text{for } n = 1, 2, \ldots. \end{aligned} \tag{4.2}$$

Now, for $r > 0$,

$$e^{-r} r^n = \frac{r^n}{e^r} \le \frac{r^n}{r^{n+1}/(n+1)!} = \frac{(n+1)!}{r},$$

For $r > 0$ and $n \in \mathbb{N}$,

$$e^r = 1 + r + \frac{r^2}{2!} + \cdots \ge \frac{r^{n+1}}{(n+1)!}.$$

so that $e^{-r} r^n \to 0$ as $r \to \infty$. Thus, by Equation (4.2), for $n = 1, 2, \ldots$, I_n exists if I_{n-1} does, and satisfies

$$I_n = n I_{n-1}.$$

Since $I_0 = 1$, we deduce that

$$I_1 = 1 \cdot I_0 = 1, \quad I_2 = 2I_1 = 2 \times 1, \quad I_3 = 3I_2 = 3 \times 2, \quad \ldots,$$

and, in general,

$$I_n = n!, \qquad \text{for } n = 0, 1, 2, \ldots.$$

So, the improper integral in Equation (4.1) exists for $n = 0, 1, 2, \ldots$, and takes the same values as the factorial function. Thus it seems at least plausible that

we should define $z!$ by

$$z! = \int_0^\infty e^{-t} t^z \, dt,$$

whenever this improper integral exists. As we shall see, this improper integral does exist for a large set of values of z and defines an analytic function. For example, we shall be able to show that

$$\left(\tfrac{1}{2}\right)! = \int_0^\infty e^{-t} t^{1/2} \, dt = \tfrac{1}{2}\sqrt{\pi} = 0.886\ldots,$$

a value which appears to be consistent with Figure 4.1.

At this point we should 'come clean' and admit that the suggestive notation $z!$ is not commonly used in complex analysis. Instead, the **gamma function** Γ is defined as follows:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt. \tag{4.3}$$

Later we shall see that the domain of Γ includes $\{z : \operatorname{Re} z > 1\}$.

Notice that, in the integral in Equation (4.3), the expression t^{z-1} appears rather than t^z. This means that

$$\Gamma(n) = (n-1)!, \qquad \text{for } n = 1, 2, \ldots, \tag{4.4}$$

which is rather unfortunate. In fact Gauss preferred the notation $\Pi(z) = \Gamma(z+1)$, so that $\Pi(n) = n!$, for $n = 0, 1, \ldots$, and this was also used by Riemann. The notation Γ was introduced by Legendre.

The Greek letter Π (capital pi) is often used to denote a product.

4.2 Differentiation under the integral sign

In order to prove that the gamma function is analytic, we shall need to be able to differentiate the improper integral in Equation (4.3) with respect to z. It is straightforward to differentiate the function

$$z \longmapsto e^{-t} t^{z-1}$$

with respect to z, for each fixed positive number t, but it is a different matter to differentiate an integral with respect to t of this function, particularly an improper one. We first prove that it is possible to differentiate with respect to z ordinary integrals of this type.

Theorem 4.1 Let $\mathcal{R}$ be a region and let K be a complex-valued function of the two variables $z \in \mathcal{R}$ and $t \in [a, b]$, such that

1. K is analytic on $\mathcal{R}$ as a function of z, for each $t \in [a, b]$;
2. K and $\partial K/\partial z$ are continuous on $[a, b]$ as functions of t, for each $z \in \mathcal{R}$;
3. for some $M > 0$,

$$|K(z,t)| \le M, \qquad \text{for } z \in \mathcal{R}, t \in [a, b].$$

Then the function

$$f(z) = \int_a^b K(z,t) \, dt \qquad (z \in \mathcal{R}) \tag{4.5}$$

is analytic on $\mathcal{R}$ and

$$f'(z) = \int_a^b \frac{\partial K}{\partial z}(z,t) \, dt, \qquad \text{for } z \in \mathcal{R}.$$

For example, $K(z,t) = e^{-t} t^{z-1}$ in Equation (4.3).

Here $\partial K/\partial z$ denotes the partial derivative of K with respect to z, keeping t fixed.

Proof Let $\alpha \in \mathcal{R}$ and choose a circle C in $\mathcal{R}$ with centre α and radius $r > 0$ such that the inside of C also lies in $\mathcal{R}$. If $\alpha + h$ lies inside C (see Figure 4.2), then we have, by Assumption 1 and Cauchy's Integral Formula,

This proof may be omitted on a first reading.

$$K(\alpha,t) = \frac{1}{2\pi i}\int_C \frac{K(z,t)}{z-\alpha}\,dz, \quad K(\alpha+h,t) = \frac{1}{2\pi i}\int_C \frac{K(z,t)}{z-(\alpha+h)}\,dz \tag{4.6}$$

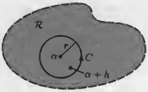


Figure 4.2

and, by Cauchy's First Derivative Formula,

$$\frac{\partial K}{\partial z}(\alpha,t) = \frac{1}{2\pi i}\int_C \frac{K(z,t)}{(z-\alpha)^2}\,dz,$$

for each $t \in [a,b]$. Hence, if f is given by Equation (4.5), then

$$\begin{aligned}
&\frac{f(\alpha+h)-f(\alpha)}{h} - \int_a^b \frac{\partial K}{\partial z}(\alpha,t)\,dt \\
&= \int_a^b \left(\frac{K(\alpha+h,t)-K(\alpha,t)}{h} - \frac{\partial K}{\partial z}(\alpha,t)\right) dt \\
&= \int_a^b \left(\frac{1}{2\pi i}\int_C \left(\frac{1}{h}\left(\frac{1}{z-\alpha-h} - \frac{1}{z-\alpha}\right) - \frac{1}{(z-\alpha)^2}\right) K(z,t)\,dz\right) dt \\
&\qquad\qquad\qquad\qquad \text{(using Equations (4.6))} \\
&= \int_a^b \left(\frac{1}{2\pi i}\int_C \frac{hK(z,t)}{(z-\alpha)^2(z-\alpha-h)}\,dz\right) dt = I(h), \text{ say,}
\end{aligned}$$

by putting the integrand over a common denominator and simplifying. To prove the desired result we need to show that this last integral $I(h)$ tends to 0 as h tends to 0. As usual, we apply the Estimation Theorem; in view of Assumption 3, we obtain

$$\begin{aligned}
\left|\frac{1}{2\pi i}\int_C \frac{hK(z,t)}{(z-\alpha)^2(z-\alpha-h)}\,dz\right| &\le \frac{1}{2\pi}\cdot\frac{|h|M}{r^2(r-|h|)}\cdot 2\pi r \\
&= \frac{|h|M}{r(r-|h|)}, \qquad \text{for } t \in [a,b].
\end{aligned}$$

For $z \in C$, $|z-\alpha| = r$ and
$$|z-\alpha-h| \ge \big||z-\alpha| - |h|\big| = r - |h|.$$

Hence

$$|I(h)| \le \frac{|h|M}{r(r-|h|)}\cdot(b-a),$$

from which it follows that $I(h) \to 0$ as $h \to 0$, as required. ■

We illustrate the use of Theorem 4.1 with a function which will have a role to play later, in the proof of Theorem 4.4.

Example 4.1

Use Theorem 4.1 to show that the function

$$f(z) = \int_0^1 \frac{e^{-z^2(t^2+1)}}{t^2+1}\,dt$$

is entire, and that

$$f'(z) = -\int_0^1 2z\,e^{-z^2(t^2+1)}\,dt, \qquad \text{for } z \in \mathbb{C}.$$

Solution

We put $K(z,t) = e^{-z^2(t^2+1)}/(t^2+1)$ and let $D = \{z : |z| < r\}$, where $r > 0$. Then

1. K is analytic on D as a function of z, for each $t \in [0,1]$, and
$$\frac{\partial K}{\partial z}(z,t) = -2ze^{-z^2(t^2+1)};$$
2. K and $\partial K/\partial z$ are continuous on $[0,1]$ as functions of t, for each $z \in D$;
3. for $|z| \le r$ and $t \in [0,1]$, we have
$$\begin{aligned} |K(z,t)| &= \left| \frac{e^{-z^2(t^2+1)}}{t^2+1} \right| \\ &\le e^{|z|^2(t^2+1)} \\ &\le e^{2r^2}. \end{aligned}$$

Remember that $|e^z| \le e^{|z|}$.

Hence, by Theorem 4.1 with $\mathcal{R} = D$, f is analytic on $D = \{z : |z| < r\}$, and
$$\begin{aligned} f'(z) &= \int_0^1 \frac{\partial K}{\partial z}(z,t)\,dt \\ &= -\int_0^1 2ze^{-z^2(t^2+1)}\,dt, \qquad \text{for } |z| < r. \end{aligned}$$

Since $r > 0$ was an arbitrary positive number, the result follows. ■

Remark The process of obtaining $f'(z)$ from $f(z)$ in this way is called 'differentiating under the integral sign'. Before using this process, it is necessary to check that Assumptions 1, 2 and 3 of Theorem 4.1 hold.

Problem 4.1

Use Theorem 4.1 to show that the function
$$f(z) = \int_{-1}^1 e^{-zt}t^2\,dt$$
is entire, and that
$$f'(z) = -\int_{-1}^1 e^{-zt}t^3\,dt, \qquad \text{for } z \in \mathbb{C}.$$

Next we use Theorem 4.1 to show that the gamma function is analytic.

> **Theorem 4.2** Let $H = \{z : \operatorname{Re} z > 1\}$. Then
> (a) Γ is analytic on H;
> (b) $\Gamma(n) = (n-1)!$, for $n > 1$;
> (c) $\Gamma(z+1) = z\Gamma(z)$, for $z \in H$.

Remark The identity in part (c) is called the **functional equation of the gamma function**.

Proof Most of this proof is devoted to part (a).

This proof may be omitted on a first reading.

(a) Consider the functions ϕ_n defined by
$$\phi_n(z) = \int_n^{n+1} K(z,t)\,dt, \qquad (z \in H_a),\ n = 0,1,2,\ldots, \tag{4.7}$$
where
$$K(z,t) = \begin{cases} e^{-t}t^{z-1}, & t > 0, \\ 0, & t = 0, \end{cases}$$
and H_a is the region
$$H_a = \{z : 1 < \operatorname{Re} z < a\}, \quad \text{for } a > 1.$$

We shall use Theorem 4.1 to show that each function ϕ_n is analytic on each region H_a, and hence on H. We then use the M-test and Weierstrass' Theorem to show that the function

$$f(z) = \sum_{n=0}^{\infty} \phi_n(z) \qquad (z \in H)$$

is analytic, and complete the proof of part (a) by showing that $\Gamma(z) = f(z)$, for $z \in H$.

First we establish the three assumptions of Theorem 4.1.

1. The function K is analytic on each H_a as a function of z, for each $t \in [n, n+1], n = 0, 1, 2, \ldots$, with

$$\frac{\partial K}{\partial z}(z,t) = \begin{cases} e^{-t}t^{z-1}\log_e t, & t > 0, \\ 0, & t = 0. \end{cases}$$

Remember that, for $t > 0$,

$$t^{z-1} = \exp((z-1)\log_e t).$$

2. The functions K and $\partial K/\partial z$ are continuous on each interval $[n, n+1], n = 0, 1, 2, \ldots$, as functions of t, for each $z \in H_a$; this is evident for $t > 0$ and at $t = 0$, we need to use the facts that

$$|K(z,t)| \leq t^{\operatorname{Re} z - 1} \to 0 \text{ as } t \to 0 \quad (\text{since } \operatorname{Re} z > 1)$$

and

$$\left|\frac{\partial K}{\partial z}(z,t)\right| \leq 1 \cdot t^{\operatorname{Re} z - 1}\log_e t \to 0 \text{ as } t \to 0.$$

The value of this limit, which is of the form

$$\lim_{\varepsilon \to 0} \varepsilon^b \log_e \varepsilon, \quad b > 0,$$

is easily deduced from the result of Problem 1.4(b)(ii).

3. For $z \in H_a$, and $t \in [n, n+1]$,

$$\begin{aligned} |K(z,t)| &\leq e^{-t}t^{\operatorname{Re} z - 1} \\ &\leq e^{-t}t^{a-1} \qquad (\text{since } z \in H_a) \\ &\leq e^{-n}(n+1)^{a-1} \qquad (\text{since } t \in [n, n+1]). \end{aligned} \tag{4.8}$$

Thus, by Theorem 4.1, each function ϕ_n is analytic on each region $H_a = \{z : 1 < \operatorname{Re} z < a\}$, and hence on $H = \{z : \operatorname{Re} z > 1\}$.

Next we apply the M-test to the sequence $\{\phi_n\}$. This is possible since if $z \in H_a$, then by *Unit B1*, Lemma 4.1, for $n = 1, 2, \ldots$,

$$\begin{aligned} |\phi_n(z)| &\leq \int_n^{n+1} |K(z,t)|\, dt \\ &\leq \int_n^{n+1} e^{-n}(n+1)^{a-1}\, dt \quad (\text{by Inequality (4.8)}) \\ &= e^{-n}(n+1)^{a-1} \\ &= M_n, \text{ say,} \end{aligned} \tag{4.9}$$

and $\sum_{n=0}^{\infty} M_n$ is convergent by the Comparison Test, for example.

Indeed, if $k = [a]$, then

$$\frac{(n+1)^{a-1}}{e^n} \leq \frac{(2n)^k}{e^n} \leq \frac{2^k(k+2)!}{n^2},$$

for $n = 1, 2, \ldots$, since

$$e^n \geq \frac{n^{k+2}}{(k+2)!},$$

and we know that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent.

Hence, by the M-test, the series of functions $\sum_{n=0}^{\infty} \phi_n$ is uniformly convergent on each H_a, so that the function

$$f(z) = \sum_{n=0}^{\infty} \phi_n(z) \qquad (z \in H) \tag{4.10}$$

is analytic on H, by Weierstrass' Theorem.

To prove that Γ is analytic on H, we need to show that

$$\Gamma(z) = \lim_{r\to\infty} \int_0^r K(z,t)\,dt = f(z), \qquad \text{for } z \in H.$$

To do this, note that if $r > 0$ and $n = [r]$, then

$$\begin{aligned}
\left| f(z) - \int_0^r K(z,t)\,dt \right| &= \left| f(z) - \left(\int_0^n K(z,t)\,dt + \int_n^r K(z,t)\,dt \right) \right| \\
&= \left| \sum_{k=n+1}^{\infty} \phi_k(z) - \int_n^r K(z,t)\,dt \right| \\
&\qquad\qquad \text{(by Equations (4.7) and (4.10))} \\
&\le \sum_{k=n+1}^{\infty} |\phi_k(z)| + \left| \int_n^r K(z,t)\,dt \right| \\
&\qquad\qquad \text{(by the Triangle Inequality)} \\
&\le \sum_{k=n}^{\infty} M_k \qquad\qquad \text{(by Equation (4.9)).}
\end{aligned}$$

Since $\sum_{n=0}^{\infty} M_n$ is convergent, we know that $\sum_{k=n}^{\infty} M_k \to 0$ as $n \to \infty$, and it follows that

$$\lim_{r\to\infty} \int_0^r K(z,t)\,dt = f(z),$$

as required. This completes the proof of part (a).

(b) We have already shown that $\Gamma(n) = (n-1)!$ (see Equation (4.4)).

(c) To prove the functional equation

$$\Gamma(z+1) = z\Gamma(z), \qquad \text{for } z \in H,$$

we note that, for $r > 0$ and $\operatorname{Re} z > 1$,

$$\begin{aligned}
\int_0^r e^{-t} t^z\,dt &= \left[-e^{-t} t^z\right]_0^r + z \int_0^r e^{-t} t^{z-1}\,dt \quad \text{(integration by parts)} \\
&= -e^{-r} r^z + z \int_0^r e^{-t} t^{z-1}\,dt.
\end{aligned}$$

Unit B1, Theorem 3.3, with the contour $[0,r]$ and parametrization
$\gamma(t) = t \quad (t \in [0,r])$.

Now, if $z = x + iy$, then

$$\left| -e^{-r} r^z \right| = \frac{r^x}{e^r} \to 0 \text{ as } r \to \infty$$

(since $r^n/e^r \to 0$ as $r \to \infty$ for any positive integer n). Hence, for $\operatorname{Re} z > 1$,

$$\begin{aligned}
\Gamma(z+1) &= \lim_{r\to\infty} \int_0^r e^{-t} t^z\,dt \\
&= \lim_{r\to\infty} \left(-e^{-r} r^z\right) + z \lim_{r\to\infty} \int_0^r e^{-t} t^{z-1}\,dt \\
&= z\Gamma(z),
\end{aligned}$$

as required. ■

Remarks

1 With more effort, it is possible to show that the gamma function, as defined by Equation (4.3) is actually analytic on $\{z : \operatorname{Re} z > 0\}$. The extra effort is required to deal with the difficulty that

$$\int_0^\infty e^{-t}t^{z-1}\,dt$$

is 'improper at 0' if $0 < \operatorname{Re} z < 1$, because then $|t^{z-1}| = t^{\operatorname{Re} z-1} \to \infty$ as $t \to 0$. However, in the next subsection we show that the gamma function can be analytically continued from $\{z : \operatorname{Re} z > 1\}$ to $\{z : \operatorname{Re} z > 0\}$ and from there to almost the whole of $\mathbb{C}$.

2 The method used in the proof of Theorem 4.2 can be adapted to prove a general result that a function f of the form

$$f(z) = \int_0^\infty K(z,t)\,dt \qquad (z \in \mathcal{R})$$

is analytic, subject to suitable hypotheses on the 'kernel function' K, and, moreover, that

$$f'(z) = \int_0^\infty \frac{\partial K}{\partial z}(z,t)\,dt, \qquad \text{for } z \in \mathcal{R}.$$

Such a result is of importance, for example, in the study of the Laplace transform of a function f, defined by

$$L_f(z) = \int_0^\infty e^{-zt}f(t)\,dt.$$

(Laplace transforms provide a method for solving certain types of ordinary and partial differential equations, which arise in many modelling applications.)

Problem 4.2

Prove the following formulas for the gamma function.

(a) $\Gamma(z) = n^z \int_0^\infty e^{-nt}t^{z-1}\,dt, \qquad$ for $n = 1, 2, \ldots,$ and $\operatorname{Re} z > 1$.

(b) $\Gamma(z) = \int_0^1 \left(\log_e \frac{1}{t}\right)^{z-1} dt, \qquad$ for $\operatorname{Re} z > 1$.

This is the form in which Euler introduced the gamma function.

4.3 Analytic continuation of the gamma function

In the previous subsection we proved that Γ is analytic on the half-plane $\{z : \operatorname{Re} z > 1\}$. Now we aim to show that Γ has an analytic continuation to almost the whole of $\mathbb{C}$.

Theorem 4.3 The gamma function has an analytic continuation Γ to $\mathbb{C} - \{0, -1, -2, \ldots\}$ with simple poles at $0, -1, -2, \ldots$, such that

$$\operatorname{Res}(\Gamma, -k) = \frac{(-1)^k}{k!}, \qquad \text{for } k = 0, 1, 2, \ldots.$$

The functional equation of the gamma function holds on $\mathbb{C} - \{0, -1, -2, \ldots\}$.

Theorem 4.3 finally answers the question posed at the beginning of this section, with $f(z) = \Gamma(z+1)$ and $\mathcal{R} = \mathbb{C} - \{-1, -2, \ldots\}$.

Remark It is conventional to refer to any analytic continuation of the function Γ, defined by Equation (4.3), by continuing to use the letter Γ and the name 'gamma function'.

Proof The key to obtaining the desired analytic continuation lies in rewriting the functional equation of the gamma function in the form

Theorem 4.2(c)

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}, \qquad \text{for } \operatorname{Re} z > 1. \tag{4.11}$$

If we now define the function Γ_0 by

$$\Gamma_0(z) = \frac{\Gamma(z+1)}{z} \qquad (\operatorname{Re} z > 0),$$

then Γ_0 is analytic on $\{z : \operatorname{Re} z > 0\}$ and agrees with Γ on $\{z : \operatorname{Re} z > 1\}$, by Equation (4.11). Thus Γ_0 is an analytic continuation of Γ to $\{z : \operatorname{Re} z > 0\}$, which we promptly rename Γ in accordance with the above remark. The process is now repeated, defining the function Γ_1 by

$$\Gamma_1(z) = \frac{\Gamma(z+1)}{z} = \frac{\Gamma(z+2)}{z(z+1)} \qquad (\operatorname{Re} z > -1, z \neq 0),$$

so that Γ_1 is analytic on $\{z : \operatorname{Re} z > -1, z \neq 0\}$ and agrees with Γ on $\{z : \operatorname{Re} z > 0\}$. Since $\Gamma(1) = \Gamma(2)/1 = 1 \neq 0$, we deduce that Γ_1 has a simple pole at 0. Renaming Γ_1 as Γ and continuing the process indefinitely, we find that Γ can be analytically continued to $\mathbb{C} - \{0, -1, -2, \ldots\}$ in such a way that the functional equation holds on this set (see Figure 4.3).

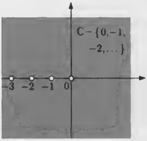


Figure 4.3

To find the residue of Γ at $-k$, for $k = 0, 1, 2, \ldots$, we note that

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \frac{\Gamma(z+2)}{z(z+1)} = \cdots = \frac{\Gamma(z+k+1)}{z(z+1)\cdots(z+k)}, \quad \text{for } z \neq 0, -1, -2, \ldots.$$

Thus, for $k = 0, 1, 2, \ldots$,

Unit C1, Theorem 1.1

$$\begin{aligned}\lim_{z\to -k}(z+k)\Gamma(z) &= \lim_{z\to -k} \frac{\Gamma(z+k+1)}{z(z+1)\cdots(z+k-1)} \\ &= \frac{\Gamma(1)}{(-k)(-k+1)\cdots(-1)} \\ &= \frac{(-1)^k}{k!},\end{aligned}$$

so that Γ has a simple pole at $-k$ with $\operatorname{Res}(\Gamma, -k) = (-1)^k/k!$. ■

Having analytically continued Γ as far as possible, the time has now come to determine the values of Γ for some points z other than $z = 1, 2, 3, \ldots$. For most points z the value of $\Gamma(z)$ can only be found by approximate methods but, as promised earlier, it is possible to evaluate $\Gamma(\frac{3}{2}) = (\frac{1}{2})! = \int_0^\infty e^{-t} t^{1/2}\, dt$.

Theorem 4.4 The value of the gamma function at $\frac{3}{2}$ is

$$\Gamma(\tfrac{3}{2}) = \tfrac{1}{2}\sqrt{\pi}.$$

Before proving this result we use it to obtain other values of Γ by means of the functional equation of the gamma function. For example, since $\Gamma(z) = \Gamma(z+1)/z$, for $z \in \mathbb{C} - \{0, -1, -2, \ldots\}$, we have

$$\Gamma(\tfrac{1}{2}) = \frac{\Gamma(\frac{3}{2})}{\frac{1}{2}} = \sqrt{\pi}. \tag{4.12}$$

Problem 4.3

Evaluate (a) $\Gamma(\frac{5}{2})$, (b) $\Gamma(-\frac{1}{2})$.

Proof of Theorem 4.4

This proof may be omitted on a first reading.

By definition

$$\Gamma(\tfrac{3}{2}) = \int_0^\infty e^{-t} t^{1/2}\, dt.$$

On substituting $t = x^2, dt = 2x\,dx$. we obtain

$$\begin{aligned}\Gamma(\tfrac{3}{2}) &= \int_0^\infty 2e^{-x^2} x^2\, dx \\ &= \lim_{r\to\infty}\left[-xe^{-x^2}\right]_0^r + \int_0^\infty e^{-x^2}\, dx \quad \text{(integration by parts)} \\ &= \int_0^\infty e^{-x^2}\, dx.\end{aligned}$$

For $r > 0$,

$$re^{-r^2} = \frac{r}{e^{r^2}} \le \frac{r}{1+r^2},$$

so

$$re^{-r^2} \to 0 \text{ as } r \to \infty.$$

We now evaluate this last integral. Since the function $f(z) = e^{-z^2}$ is entire, it has a primitive F on $\mathbb{C}$, by the Primitive Theorem. By the proof of the Primitive Theorem, F is given by

Unit B2, Theorem 5.3

$$F(z) = \int_0^z e^{-w^2}\, dw,$$

where this notation means that the integral is taken along any grid path from 0 to z. By the Contour Independence Theorem (*Unit B2*, Theorem 1.3), we can integrate instead along the line segment from 0 to z (see Figure 4.4). Using the parametrization $\gamma(t) = zt$ $(t \in [0,1])$, for which $\gamma'(t) = z$, we obtain

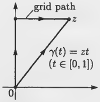


Figure 4.4

$$F(z) = \int_0^1 e^{-z^2t^2} z\, dt.$$

Next, put $g(z) = (F(z))^2$. Then

You will see the usefulness of g shortly.

$$\begin{aligned}g'(z) &= 2F(z)F'(z) \qquad \text{(Chain Rule)} \\ &= 2e^{-z^2}\int_0^1 e^{-z^2t^2} z\, dt \qquad (F'(z) = f(z) = e^{-z^2}) \\ &= \int_0^1 2ze^{-z^2(t^2+1)}\, dt \\ &= -\int_0^1 \frac{\partial}{\partial z}\left(\frac{e^{-z^2(t^2+1)}}{t^2+1}\right) dt \\ &= -\frac{d}{dz}\int_0^1 \frac{e^{-z^2(t^2+1)}}{t^2+1}\, dt\end{aligned}$$

(see Theorem 4.1 and Example 4.1). Thus, if

$$h(z) = \int_0^1 \frac{e^{-z^2(t^2+1)}}{t^2+1}\, dt \qquad (z \in \mathbb{C}),$$

then

$$g'(z) + h'(z) = 0, \qquad \text{for } z \in \mathbb{C},$$

so that $g + h$ is constant on $\mathbb{C}$. Hence, for $z \in \mathbb{C}$,

$$g(z) + h(z) = g(0) + h(0) = 0 + \int_0^1 \frac{1}{t^2+1}\, dt = \frac{\pi}{4}. \tag{4.13}$$

$$g(0) = (F(0))^2 = \left(\int_0^1 0\, dt\right)^2 = 0.$$

Now,

$$0 \le \frac{e^{-x^2(t^2+1)}}{t^2+1} \le e^{-x^2}, \qquad \text{for } x \ge 0, t \ge 0,$$

so that

$$0 \le \int_0^1 \frac{e^{-x^2(t^2+1)}}{t^2+1}\, dt \le \int_0^1 e^{-x^2}\, dt = e^{-x^2}, \qquad \text{for } x \ge 0.$$

Hence $h(x) \to 0$ as $x \to \infty$, and so, using Equation (4.13),

$$g(x) = \frac{\pi}{4} - h(x) \to \frac{\pi}{4} \qquad \text{as } x \to \infty.$$

Thus, finally,

$$\int_0^x e^{-t^2}\,dt = F(x) = \sqrt{g(x)} \to \sqrt{\pi/4} = \tfrac{1}{2}\sqrt{\pi}, \text{ as } x \to \infty,$$

so that

$$\Gamma(\tfrac{3}{2}) = \int_0^\infty e^{-t^2}\,dt = \tfrac{1}{2}\sqrt{\pi}. \quad \blacksquare$$

Remarks

1 The integral $\int_0^\infty e^{-x^2}\,dx$ appearing in the proof of Theorem 4.4 is of fundamental importance in statistics. Indeed, the function

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad (x \in \mathbb{R})$$

is the probability density function of the standard normal distribution, with mean 0 and standard deviation 1. The factor $1/\sqrt{2\pi}$ is needed in order to ensure that

$$\int_{-\infty}^\infty f(x)\,dx = 1.$$

2 There are other ways of evaluating $\int_0^\infty e^{-x^2}\,dx$. Perhaps the simplest is based on *double integrals* and we sketch the argument here in case you are familiar with these:

$$\begin{aligned}
\left(\int_0^\infty e^{-x^2} dx\right)^2 &= \left(\int_0^\infty e^{-x^2} dx\right)\left(\int_0^\infty e^{-y^2} dy\right) \\
&= \int_0^\infty \int_0^\infty e^{-x^2} e^{-y^2}\,dxdy \\
&= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dxdy \\
&= \int_0^\infty \int_0^{\pi/2} e^{-r^2} r\,d\theta dr \\
&= \frac{\pi}{2}\int_0^\infty e^{-r^2} r\,dr \\
&= \frac{\pi}{2}\left[-\frac{1}{2}e^{-r^2}\right]_0^\infty = \frac{\pi}{4},
\end{aligned}$$

Substitute
$x = r\cos\theta$,
$y = r\sin\theta$,
and replace
$dxdy$ by $r\,d\theta dr$.

so that

$$\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi/4} = \tfrac{1}{2}\sqrt{\pi}.$$

3 This is a good moment to discuss briefly why an integral such as $\int_0^\infty e^{-x^2}\,dx$ cannot be evaluated by using the Fundamental Theorem of Calculus. You have perhaps tried in vain to find a function (involving combinations and compositions of elementary functions) which, when differentiated, gives $x \longmapsto e^{-x^2}$. Probably you are convinced that no such function exists, but how can such a result be *proved*? In fact the general question of which functions can be 'integrated in finite terms' (that is, have a primitive among the elementary functions) was studied systematically by Liouville. He developed a theory of integration in finite terms, involving a careful study of the singularities of the given function, which was largely forgotten until 1970, when R. H. Risch published an algorithm (based on Liouville's ideas) for carrying out this process. The algorithm produces the desired primitive if this is possible and shows that no such primitive exists otherwise. (Needless to say, the algorithm shows that $x \longmapsto e^{-x^2}$ does not have an elementary primitive!) Since then the algorithm has been greatly extended and is widely used for symbolic integration in computer packages.

This is the Liouville of Liouville's Theorem, which you met in *Unit B2*.

4.4 Calculating the gamma function

We start this final subsection by giving an alternative formula for defining the gamma function:

This subsection is intended for reading only.

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\,n^z}{z(z+1)\cdots(z+n)}, \qquad \text{for } z \in \mathbb{C} - \{0,-1,-2,\ldots\}. \qquad (4.14)$$

This formula has several advantages over Equation (4.3). It is valid for all z, apart from the poles of Γ, and it can be used to calculate approximate values of $\Gamma(z)$. For example,

$$\Gamma(\tfrac{1}{2}) = \lim_{n\to\infty} \frac{n!\,n^{1/2}}{\frac{1}{2}(\frac{1}{2}+1)\cdots(\frac{1}{2}+n)}.$$

Unfortunately, the sequence $\{(n!\,n^z)/(z(z+1)\cdots(z+n))\}$ converges extremely slowly, as the values in this table (for $z=\frac{1}{2}$) indicate.

n	1	2	3	4	5	6	...	100
$\dfrac{n!\,n^{1/2}}{\frac{1}{2}\left(\frac{1}{2}+1\right)\cdots\left(\frac{1}{2}+n\right)}$	1.33	1.51	1.58	1.63	1.65	1.67		1.766

These values should be compared with the known value $\Gamma(\frac{1}{2}) = \sqrt{\pi} = 1.772\ldots$.

Nevertheless, with the help of a computer, Equation (4.14) can be used to obtain the following graph $y = \Gamma(x)$.

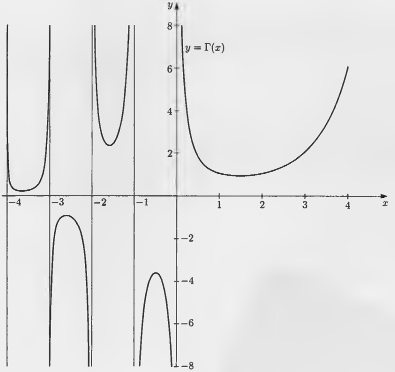


Figure 4.5

Perhaps this graph goes some way to explaining why the function Γ was preferred to Gauss' function Π, whose graph would be the graph of Γ shifted one unit to the left.

The graph shows quite clearly the simple poles of Γ at $0, -1, -2, -3, -4, \ldots$.

We do not have space to give a full proof that Equations (4.3) and (4.14) define the same function for $\operatorname{Re} z > 1$, but here is a sketch of the argument. Starting from the equation

$$e^{-t} = \lim_{n\to\infty} \left(1 - \frac{t}{n}\right)^n,$$

proved in courses on real analysis, it can be shown that, for $\operatorname{Re} z > 1$,

$$\int_0^\infty e^{-t} t^{z-1}\,dt = \lim_{n\to\infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}\,dt.$$

Now

$$\int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\,dt = n^z \int_0^1 (1-s)^n s^{z-1}\,ds \qquad (t = ns, dt = n\,ds)$$

$$= n^z \cdot \frac{n}{z}\int_0^1 (1-s)^{n-1} s^z\,ds \quad \text{(integration by parts)}$$

$$= n^z \cdot \frac{n(n-1)}{z(z+1)}\int_0^1 (1-s)^{n-2} s^{z+1}\,ds$$

$$= \cdots$$

$$= n^z \cdot \frac{n(n-1)\cdots 2\cdot 1}{z(z+1)\cdots(z+n-1)}\int_0^1 s^{z+n-1}\,ds$$

$$\text{(repeated integration by parts)}$$

$$= \frac{n!\,n^z}{z(z+1)\cdots(z+n)},$$

so that, for $\operatorname{Re} z > 1$,

$$\int_0^\infty e^{-t}t^{z-1}dt = \lim_{n\to\infty}\frac{n!\,n^z}{z(z+1)\cdots(z+n)}.$$

Finally, we point out some of the many other remarkable properties of the gamma function. For example, it can be shown that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \qquad (4.15)$$

from which it follows that Γ has no zeros, and also, again, that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Also, we note that the calculation of $\Gamma(z)$ is greatly facilitated by a complex form of Stirling's formula: $n! \sim \sqrt{2\pi n}(n/e)^n$. One version of this formula states that

$$\Gamma(z) \cong \sqrt{\frac{2\pi}{z}}\left(\frac{z}{e}\right)^z, \qquad \text{for } |\operatorname{Arg} z| \le \frac{\pi}{4}.$$

If $\{a_n\}$ and $\{b_n\}$ are sequences of positive terms, then

$a_n \sim b_n$

means that

$\frac{a_n}{b_n} \to 1$ as $n \to \infty$.

To give you some idea of the behaviour of the gamma function for complex values of z, we give in Figure 4.6 the graph of the surface $s = |\Gamma(z)|$, for $-4 \le \operatorname{Re} z, \operatorname{Im} z \le 4$, produced using *Mathematica*. Notice the simple poles on the x-axis.

Mathematica (Wolfram Research Inc.) is a software package.

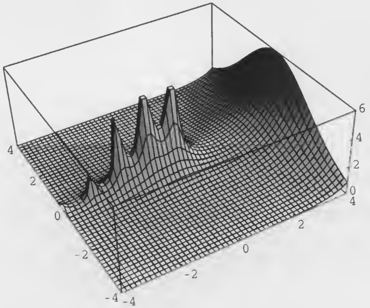


Figure 4.6 $s = |\Gamma(z)|$

5 RIEMANN'S LEGACY

After working through this section, you should be able to:

(a) appreciate why complex analysis has a role to play in number theory.

This section is intended for reading only.

5.1 The Prime Number Theorem

As long ago as 1740, Euler was aware of the connection between the zeta function

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \cdots \qquad (\operatorname{Re} z > 1)$$

As noted earlier, Euler considered $\zeta(z)$ only for z real and greater than 1.

and the sequence of prime numbers:

$$2, 3, 5, 7, 11, 13, 17, 19, \ldots .$$

To see this connection, notice that

$$\frac{1}{2^z}\zeta(z) = \frac{1}{2^z} + \frac{1}{4^z} + \frac{1}{6^z} + \cdots ,$$

so that

$$\left(1 - \frac{1}{2^z}\right)\zeta(z) = 1 + \frac{1}{3^z} + \frac{1}{5^z} + \frac{1}{7^z} + \cdots .$$

Similarly,

$$\left(1 - \frac{1}{2^z}\right)\left(1 - \frac{1}{3^z}\right)\zeta(z) = 1 + \frac{1}{5^z} + \frac{1}{7^z} + \frac{1}{11^z} + \frac{1}{13^z} + \cdots ,$$

and, continuing indefinitely, we obtain

$$\left(\left(1 - \frac{1}{2^z}\right)\left(1 - \frac{1}{3^z}\right)\left(1 - \frac{1}{5^z}\right)\cdots\right)\zeta(z) = 1;$$

that is,

$$\zeta(z) = \left(\left(1 - \frac{1}{2^z}\right)\left(1 - \frac{1}{3^z}\right)\left(1 - \frac{1}{5^z}\right)\cdots\right)^{-1}, \qquad \text{for } \operatorname{Re} z > 1. \qquad (5.1)$$

Here the expression in brackets represents the infinite product of all factors of the form $(1 - 1/p^z)$, in which p is a prime number.

Riemann saw that Formula (5.1) could be used to bring techniques and results from complex analysis to bear upon a long-standing problem concerning the distribution of prime numbers. At the end of the 18th century, both Gauss and Legendre had observed, by calculation, that the sequence of prime numbers thins out in a way that, although locally erratic, can be described rather precisely in the long term. Their observation concerns the counting function

$$\pi(x) = \text{ the number of primes less than } x \qquad (x > 0),$$

some values of which are recorded below, along with the corresponding values of $\log_e$ for comparison.

x	10	10^2	10^3	10^6
$\pi(x)$	4	25	168	78498
$\log_e x$	2.303	4.605	6.908	13.816

As you would expect, it appears that $\pi(x)/x \to 0$ as $x \to \infty$, but on closer inspection it also appears that

$$\pi(x)\log_e x \cong x.$$

This led to the conjecture that

$$\lim_{x\to\infty} \frac{\pi(x)\log_e x}{x} = 1,$$

a result which is now called the **Prime Number Theorem**.

Some progress towards the Prime Number Theorem had been made by Riemann's time, but it was still very far from being proved. In a remarkable paper 'Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse' ('On the number of primes less than a given magnitude') in 1859, Riemann showed that a proof of the Prime Number Theorem could be given if certain properties of the zeta function could be established. These properties have to do with the location of the zeros of the function ζ. Now Equation (5.1) shows that the zeta function has no zeros for $\operatorname{Re} z > 1$, but Riemann found an analytic continuation of ζ to the whole of $\mathbb{C} - \{1\}$ and it was the zeros of this analytic continuation which concerned him.

When discussing the zeta function Riemann used the notation $s = \sigma + i\tau$ for the complex variable, and to this day many texts use $\zeta(s)$ rather than $\zeta(z)$.

5.2 The Riemann hypothesis

Riemann derived his analytic continuation of the zeta function by starting from the gamma function, as follows. For $\operatorname{Re} z > 1$,

$$\begin{aligned}\Gamma(z) &= \int_0^\infty e^{-t} t^{z-1}\, dt \\ &= \int_0^\infty e^{-nu} (nu)^{z-1} n\, du \quad (t = nu, dt = n\, du) \\ &= n^z \int_0^\infty e^{-nu} u^{z-1}\, du, \qquad \text{for } n = 1, 2, \ldots .\end{aligned}$$

Hence, writing the integration variable as t, we deduce that

$$\frac{\Gamma(z)}{n^z} = \int_0^\infty e^{-nt} t^{z-1}\, dt, \qquad \text{for } n = 1, 2, \ldots,$$

and then that

$$\begin{aligned}\Gamma(z) \sum_{n=1}^\infty \frac{1}{n^z} &= \sum_{n=1}^\infty \int_0^\infty e^{-nt} t^{z-1}\, dt \\ &= \int_0^\infty \left(\sum_{n=1}^\infty \left(e^{-t}\right)^n \right) t^{z-1}\, dt \\ &= \int_0^\infty \frac{t^{z-1}}{e^t - 1}\, dt,\end{aligned}$$

Interchanging the $\int$ and $\sum$ symbols here needs some justification!

by summing the geometric series. Since the gamma function has no zeros, it follows that

$$\zeta(z) = \sum_{n=1}^\infty \frac{1}{n^z} = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^t - 1}\, dt, \qquad \text{for } \operatorname{Re} z > 1. \tag{5.2}$$

By manipulating the integral in Equation (5.2), Riemann obtained the representation

$$\zeta(z) = \frac{1}{\Gamma(z)\left(1 - e^{2\pi i z}\right)} \int_C \frac{\exp((z-1)\operatorname{Log}_{2\pi} w)}{e^w - 1}\, dw, \qquad \text{for } \operatorname{Re} z > 1, (5.3)$$

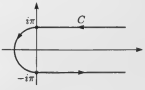


Figure 5.1

where C is an infinite contour such as the one in Figure 5.1. Now it is not hard to see that the contour integral in Equation (5.3) defines a function which is analytic on the whole of $\mathbb{C}$ and is non-zero at $z = 1$. Since the function

$$z \longmapsto \Gamma(z)\left(1 - e^{2\pi i z}\right)$$

has removable singularities at $0, -1, -2, \ldots$, we deduce that Equation (5.3) yields an analytic continuation of ζ to the whole of $\mathbb{C} - \{1\}$, with a simple pole at 1 (arising from the term $\left(1 - e^{2\pi i z}\right)$). Riemann showed that this analytic continuation satisfies the remarkable functional equation

The poles of Γ are 'cancelled' by the zeros of $z \longmapsto 1 - e^{2\pi i z}$.

$$(2\pi)^z \zeta(1 - z) = 2\Gamma(z) \cos\left(\tfrac{1}{2}\pi z\right) \zeta(z).$$

Since $\zeta(1 - z)$ is finite and non-zero for $\operatorname{Re}(1 - z) > 1$, it follows that

$$\Gamma(z) \cos\left(\tfrac{1}{2}\pi z\right) \zeta(z) \text{ is finite and non-zero for } \operatorname{Re} z < 0.$$

Thus the zeros of ζ in $\{z : \operatorname{Re} z < 0\}$ can only lie at the poles of Γ. But Γ has simple poles at $-1, -2, -3, \ldots$, and $z \longmapsto \cos\left(\frac{1}{2}\pi z\right)$ has simple zeros which 'cancel' these only at $-1, -3, -5, \ldots$. Hence ζ must have simple zeros at $-2, -4, -6, \ldots$ and at no other points of $\{z : \operatorname{Re} z < 0\}$. It is traditional to refer to these zeros as the **trivial zeros** of the zeta function (because we know all about them). Any other zeros of ζ must lie in $\{z : 0 \le \operatorname{Re} z \le 1\}$, the so-called **critical strip** (see Figure 5.2). Riemann showed that the Prime Number Theorem could be proved if there are not too many zeros of ζ near the edges of the critical strip, and he made the startling conjecture that in fact all the zeros of ζ in the critical strip lie on the **critical line** $\{z : \operatorname{Re} z = \frac{1}{2}\}$. (Figure 5.2 shows the two zeros in the critical strip which are closest to the real axis. The values are correct to two decimal places.) This conjecture is now known as the **Riemann hypothesis** and it is still unresolved at the time of writing (1993), despite the efforts of some of the best mathematicians since Riemann's day.

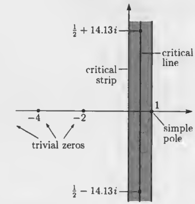


Figure 5.2

No one is sure on what evidence Riemann made his conjecture, but extensive numerical investigations have detected more than a billion zeros of ζ in the critical strip, *all* lying on $\{z : \operatorname{Re} z = \frac{1}{2}\}$. The first few of these, and the simple pole at 0, are visible in Figure 5.3, which is a plot of the surface $s = |\zeta(z)|$, for $-1 \le \operatorname{Re} z \le 8, -25 \le \operatorname{Im} z \le 25$, produced using *Mathematica.*

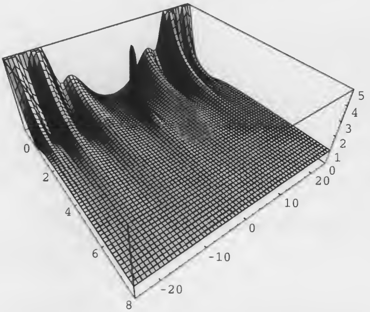


Figure 5.3 $s = |\zeta(z)|$

G. H. Hardy proved in 1924 that infinitely many zeros of ζ lie on the critical line, but this does not prevent some others lying off it. Better still, N. Levinson proved in 1974 that, in a certain sense, at least one third of the zeros in the critical strip lie on the critical line.

The Prime Number Theorem itself was proved in 1896 by J. Hadamard and C. de la Vallée Poussin (independently), using the complex analytic approach suggested by Riemann, but without needing the full strength of the Riemann hypothesis. Since then several other proofs have been found, including one which does not require complex analysis, but the simplest so far was given in 1980 by D. Newman. This proof again used the zeta function, but only needed the elementary fact that ζ is zero-free on $\{z : \operatorname{Re} z \ge 1\}$.

In spite of this, interest in the Riemann hypothesis remains strong, partly because many results in number theory have been proved on the assumption that the Riemann hypothesis is true. Proving such results may seem pointless if we do not know whether the Riemann hypothesis is true, but there is an ulterior motive — should any of these results lead to a contradiction, then the Riemann hypothesis would have to be false!

EXERCISES

Section 1

Exercise 1.1 Show that each of the following pairs of analytic functions f and g are direct analytic continuations of each other.

(a) $f(z) = (z+1) + (z+1)^2 + (z+1)^3 + \cdots \qquad (|z+1| < 1)$

$g(z) = -\dfrac{1}{z} - 1 \qquad (z \in \mathbb{C} - \{0\})$

(b) $f(z) = (z+1) + \frac{1}{2}(z+1)^2 + \frac{1}{3}(z+1)^3 + \cdots \qquad (|z+1| < 1)$

$g(z) = i\pi - \operatorname{Log}_{2\pi}(z) \qquad (z \in \mathbb{C}_{2\pi})$

(*Hint*: In part (b), it helps to use the fact that if $\operatorname{Re} z > 0$, then

$\operatorname{Arg}(z) + \pi = \operatorname{Arg}_{2\pi}(-z)$.)

Exercise 1.2 Let

$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \qquad (|z| < 1).$$

(a) Prove that

$$f'(z) = \frac{-\operatorname{Log}(1-z)}{z}, \qquad \text{for } 0 < |z| < 1.$$

(b) Deduce that the function

$$g(z) = \begin{cases} \dfrac{-\operatorname{Log}(1-z)}{z}, & z \in \mathbb{C} - \{x \in \mathbb{R} : x = 0 \text{ or } x \geq 1\}, \\ 1, & z = 0, \end{cases}$$

is a direct analytic continuation of f' from $\{z : |z| < 1\}$ to $\mathbb{C} - \{x \in \mathbb{R} : x \geq 1\}$.

(c) Use the Primitive Theorem (*Unit B2*, Theorem 5.3) to show that there is a direct analytic continuation of f from $\{z : |z| < 1\}$ to a larger region.

Exercise 1.3 Use the approach in the audio tape to evaluate the following improper integral.

$$\int_0^{\infty} \frac{\log_e t}{(1+t^2)^2}\, dt$$

Exercise 1.4 Use Theorem 1.1 to evaluate the following improper integral.

$$\int_0^{\infty} \frac{t^{3/2}}{(t^2+1)(t-1)}\, dt$$

Section 2

Exercise 2.1 Show that the functions

$$f(z) = \sum_{n=0}^{\infty} z^n \quad (|z| < 1) \qquad \text{and} \qquad g(z) = -\sum_{n=1}^{\infty} z^{-n} \quad (|z| > 1)$$

are indirect analytic continuations of each other.

Exercise 2.2 Use the result of Exercise 1.1(b) to show that the functions

$$f(z) = (z-1) - \tfrac{1}{2}(z-1)^2 + \tfrac{1}{3}(z-1)^3 - \cdots \qquad (|z-1| < 1)$$

and

$$g(z) = i\pi - (z+1) - \tfrac{1}{2}(z+1)^2 - \tfrac{1}{3}(z+1)^3 - \cdots \qquad (|z+1| < 1)$$

are indirect analytic continuations of each other.

Section 3

Exercise 3.1 Prove that the sequence of functions

$$f_n(z) = z + \frac{z^2}{n}, \qquad n = 1, 2, \ldots,$$

converges uniformly on each closed disc of the form $\{z : |z| \le r\}$, where $r > 0$.

Exercise 3.2 Prove that the series

$$\sum_{n=1}^{\infty} \frac{z^n}{1+z^n}$$

is uniformly convergent on each closed disc of the form $E = \{z : |z| \le r\}$, where $0 < r < 1$. Deduce that the sum function f is analytic on $\{z : |z| < 1\}$, and write down a formula for f'.

Exercise 3.3 Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{z^2 - n^2}$$

is uniformly convergent on each closed disc in $\mathbb{C} - \mathbb{Z}$ and hence defines an analytic function, f say, on $\mathbb{C} - \mathbb{Z}$. Write down a formula for the derivative f'.

(*Hint*: To prove that the series is uniformly convergent on a closed disc $E = \{z : |z| \le r\}$, it is sufficient to prove that the series

$$\sum_{n \ge 2r} \frac{1}{z^2 - n^2}$$

is uniformly convergent on E.)

Section 4

Exercise 4.1 Prove that the function

$$f(z) = \int_{-1}^{1} \sin(zt^2)\, dt$$

is entire, and obtain a formula for the derivative of f.

Exercise 4.2 Evaluate each of the following expressions.

(a) $\Gamma(-3/2)$ (b) $\dfrac{\Gamma(-3+i)}{\Gamma(i)}$

Exercise 4.3 The formula

$$V_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)}\, r^n, \qquad n = 1, 2, \ldots,$$

gives the n-dimensional volume of the ball of radius r in $\mathbb{R}^n$:

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + x_2^2 + \cdots + x_n^2 \le r^2\}.$$

Verify that
(a) $V_1 = 2r$; (b) $V_2 = \pi r^2$; (c) $V_3 = \frac{4}{3}\pi r^3$; (d) $V_4 = \frac{1}{2}\pi^2 r^4$.

Exercise 4.4 A remarkable identity, due to Euler, is

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\, dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Euler called this function of α and β the **beta function**.

Verify that this identity holds for $\alpha = \beta = \frac{1}{2}$ (by using the substitution $t = \sin^2\theta$ in the integral on the left).

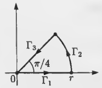
Exercise 4.5 The Fresnel integrals

$$\int_0^\infty \cos(t^2)\,dt \quad \text{and} \quad \int_0^\infty \sin(t^2)\,dt$$

are important in optics. Evaluate these integrals by considering the contour integral

$$\int_\Gamma e^{-z^2}\,dz,$$

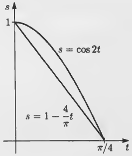
where $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$ is shown in the figure.



(*Hint*: On Γ_2 use the parametrization $\gamma(t) = re^{it}$ $(t \in [0, \pi/4])$ and estimate the integral along Γ_2 by using *Unit B1*, Lemma 4.1 and the inequality

$$\cos 2t \geq 1 - \frac{4}{\pi}t, \qquad \text{for } 0 \leq t \leq \pi/4,$$

illustrated in the second figure.)



SOLUTIONS TO THE PROBLEMS

Section 1

1.1 (a) Since $\sum_{n=0}^{\infty}(2z)^n$ is a geometric series with sum

$$\frac{1}{1-2z}, \quad \text{for } |z| < \tfrac{1}{2},$$

we deduce that the function

$$g(z) = \frac{1}{1-2z} \quad (z \in \mathbb{C} - \{\tfrac{1}{2}\})$$

is an analytic extension of f to $\mathbb{C} - \{\frac{1}{2}\}$.

(b) Since

$$\begin{aligned}\operatorname{Log} z &= \operatorname{Log}(1 + (z-1)) \\ &= (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \cdots \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(z-1)^n, \quad \text{for } |z-1| < 1,\end{aligned}$$

we deduce that the function

$$g(z) = \operatorname{Log} z \qquad (z \in \mathbb{C} - \{x \in \mathbb{R} : x \le 0\})$$

is an analytic extension of f to $\mathbb{C} - \{x \in \mathbb{R} : x \le 0\}$.

1.2 (a) Here $\mathcal{R} = \{z : |z| < 1\}$ and $\mathcal{S} = \mathbb{C} - \{1\}$. Since

$$\begin{aligned}(1-z)^{-2} &= 1 + 2z + 3z^2 + \cdots \quad \text{(Binomial series)} \\ &= \sum_{n=1}^{\infty} nz^{n-1}, \quad \text{for } |z| < 1,\end{aligned}$$

f and g agree on the region $\mathcal{T} = \{z : |z| < 1\} \subseteq \mathcal{R} \cap \mathcal{S}$. Hence we deduce that f and g are direct analytic continuations of each other.

(b) Here $\mathcal{R} = \mathbb{C}_{2\pi}$ and $\mathcal{S} = \mathbb{C}_{3\pi/2}$. Since f and g agree on the region $\mathcal{T} = \{z : \operatorname{Re} z < 0\} \subseteq \mathcal{R} \cap \mathcal{S}$, we deduce that f and g are direct analytic continuations of each other.

1.3 First note that f is the principal square root function:

$$f(z) = \sqrt{z} = \exp(\tfrac{1}{2}\operatorname{Log}_\pi(z)) \quad (z \in \mathbb{C}_\pi),$$

since $\operatorname{Log} = \operatorname{Log}_\pi$. Let us then consider the function

$$g(z) = \exp(\tfrac{1}{2}\operatorname{Log}_{3\pi/2}(z)) \quad (z \in \mathbb{C}_{3\pi/2}),$$

chosen because $z \longmapsto \operatorname{Log}_{3\pi/2}(z)$ is a direct analytic continuation of $z \longmapsto \operatorname{Log}_\pi(z)$ (see Example 1.2(b)). Since

$$f(z) = g(z), \quad \text{for } z \in \mathcal{T},$$

where $\mathcal{T} = \{z : \operatorname{Re} z > 0\} \subseteq \mathbb{C}_\pi \cap \mathbb{C}_{3\pi/2}$, we deduce that g is a direct analytic continuation of f from $\mathbb{C}_\pi$ to $\mathbb{C}_{3\pi/2}$.

1.4 (a) Since

$$\frac{1}{t} \le \frac{1}{\sqrt{t}}, \quad \text{for } t \ge 1,$$

it follows from the Monotonicity Inequality (*Unit B1*, Frame 6) that if $r > 1$ then

$$\int_1^r \frac{1}{t}\,dt \le \int_1^r \frac{1}{\sqrt{t}}\,dt.$$

Evaluating each side of this inequality, we obtain

$$[\log_e t]_1^r \le [2\sqrt{t}]_1^r,$$

that is,

$$\log_e r \le 2\sqrt{r} - 2 \le 2\sqrt{r}, \quad \text{for } r > 1. \tag{1}$$

(b) (i) Using Inequality (1), we have

$$0 < \frac{\log_e r}{r} \le \frac{2}{\sqrt{r}}, \quad \text{for } r > 1.$$

Since $\lim_{r\to\infty} \frac{2}{\sqrt{r}} = 0$, it follows that

$$\lim_{r\to\infty} \frac{\log_e r}{r} = 0.$$

(ii) Let $\varepsilon = 1/r$. Then, from Inequality (1),

$$\log_e 1/\varepsilon \le 2\sqrt{1/\varepsilon}, \quad \text{for } 0 < \varepsilon < 1.$$

Hence, multiplying though by ε, we obtain

$$\varepsilon(\log_e 1 - \log_e \varepsilon) \le 2\sqrt{\varepsilon}, \quad \text{for } 0 < \varepsilon < 1,$$

so that

$$-2\sqrt{\varepsilon} \le \varepsilon \log_e \varepsilon < 0, \quad \text{for } 0 < \varepsilon < 1.$$

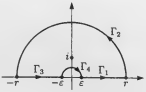
Since $\lim_{\varepsilon\to 0}(-2\sqrt{\varepsilon}) = 0$, it follows that

$$\lim_{\varepsilon \to 0} \varepsilon \log_e \varepsilon = 0.$$

1.5 (a) Consider the contour integral

$$I = \int_\Gamma \frac{\operatorname{Log}_{3\pi/2}(z)}{z^2+1}\,dz,$$

where $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$ is the contour shown in the figure (with $r > 1$ and $0 < \varepsilon < 1$).



(b) If $f(z) = (\operatorname{Log}_{3\pi/2}(z))/(z^2+1)$, then, by the Residue Theorem and the g/h Rule with $g(z) = \operatorname{Log}_{3\pi/2}(z)$, $h(z) = z^2+1$ and $h'(z) = 2z$,

$$\begin{aligned}I &= 2\pi i \times \operatorname{Res}(f, i) \\ &= 2\pi i \times \frac{\operatorname{Log}_{3\pi/2}(i)}{2 \times i} \\ &= \pi(\log_e |i| + i\operatorname{Arg}_{3\pi/2}(i)) \\ &= \pi(0 + i\pi/2) = \tfrac{1}{2}i\pi^2.\end{aligned}$$

(c) Splitting up the integral gives

$$\begin{aligned}I &= \int_{\Gamma_1} \cdots\,dz + \int_{\Gamma_2} \cdots\,dz + \int_{\Gamma_3} \cdots\,dz + \int_{\Gamma_4} \cdots\,dz \\ &= \tfrac{1}{2}i\pi^2,\end{aligned} \tag{2}$$

where

$$\int_{\Gamma_1} \cdots\,dz = \int_\varepsilon^r \frac{\log_e t}{t^2+1}\,dt$$

and

$$\begin{aligned}\int_{\Gamma_3} \cdots\,dz &= \int_{-r}^{-\varepsilon} \frac{\log_e |t| + i\pi}{t^2+1}\,dt \\ &= \int_\varepsilon^r \frac{\log_e t}{t^2+1}\,dt + i\pi \int_\varepsilon^r \frac{1}{t^2+1}\,dt,\end{aligned}$$

since the integrand is an even function.

(d) We now estimate the integrals along Γ_2 and Γ_4.

On Γ_2, $|z| = r$ so, by the Triangle Inequality,

$$|z^2 + 1| \geq r^2 - 1$$

and

$$\begin{aligned} |\operatorname{Log}_{3\pi/2}(z)| &= |\log_e |z| + i \operatorname{Arg}_{3\pi/2}(z)| \\ &\leq \log_e r + \pi. \end{aligned}$$

Hence, by the Estimation Theorem,

$$\left| \int_{\Gamma_2} \frac{\operatorname{Log}_{3\pi/2}(z)}{z^2+1} dz \right| \leq \frac{\log_e r + \pi}{r^2 - 1} \times \pi r, \quad \text{for } r > 1. \quad (3)$$

Similarly

$$\left| \int_{\Gamma_4} \frac{\operatorname{Log}_{3\pi/2}(z)}{z^2+1} dz \right| \leq \frac{\pi - \log_e \varepsilon}{1 - \varepsilon^2} \times \pi \varepsilon, \quad \text{for } 0 < \varepsilon < 1. (4)$$

(e) We now let $r \to \infty$ and $\varepsilon \to 0$ in Equation (2). Then

$$\int_{\Gamma_1} \cdots dz = \int_\varepsilon^r \frac{\log_e t}{t^2+1} dt \to \int_0^\infty \frac{\log_e t}{t^2+1} dt,$$

$$\begin{aligned} \int_{\Gamma_3} \cdots dz &= \int_\varepsilon^r \frac{\log_e t}{t^2+1} dt + i\pi \int_\varepsilon^r \frac{1}{t^2+1} dt, \\ &\to \int_0^\infty \frac{\log_e t}{t^2+1} dt + i\pi \int_0^\infty \frac{1}{t^2+1} dt, \end{aligned}$$

provided that these limits exist. Their existence follows because

$$\int_{\Gamma_2} \frac{\operatorname{Log}_{3\pi/2}(z)}{z^2+1} dz \to 0 \text{ as } r \to \infty$$

(by Inequality (3) and Problem 1.4(b)) and

$$\int_{\Gamma_4} \frac{\operatorname{Log}_{3\pi/2}(z)}{z^2+1} dz \to 0 \text{ as } \varepsilon \to 0$$

(by Inequality (4) and Problem 1.4(b)).

Equation (2) becomes

$$2 \int_0^\infty \frac{\log_e t}{t^2+1} dt + i\pi \int_0^\infty \frac{1}{t^2+1} dt = \tfrac{1}{2} i \pi^2.$$

Hence, equating real parts, we obtain

$$\int_0^\infty \frac{\log_e t}{t^2+1} dt = 0.$$

(Equating imaginary parts gives $\displaystyle\int_0^\infty \frac{1}{t^2+1} dt = \frac{\pi}{2}$.)

Remark The integral $\displaystyle\int_0^\infty \frac{\log_e t}{t^2+1} dt$ can also be evaluated by using the substitution $u = 1/t$, $du = (-1/t^2)\,dt$ from which $dt = (-1/u^2)\,du$. Then

$$\begin{aligned} \int_0^\infty \frac{\log_e t}{t^2+1} dt &= \int_\infty^0 \frac{-\log_e u}{(1/u^2)+1} \left(-\frac{1}{u^2}\right) du \\ &= -\int_0^\infty \frac{\log_e u}{u^2+1} du, \end{aligned}$$

and so

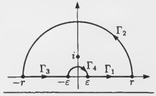
$$\int_0^\infty \frac{\log_e t}{t^2+1} dt = 0.$$

This method is much simpler, for those who spot it. The power of the contour integration method is that it applies to whole classes of improper integrals.

1.6 **(a)** Consider the contour integral

$$I = \int_\Gamma \frac{\exp(-b \operatorname{Log}_{3\pi/2}(z))}{z^2+1} dz,$$

where $b = 1 - a$, so that $0 < b < 1$, and Γ is the contour $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$ shown in the figure (with $r > 1$ and $0 < \varepsilon < 1$). (Frame 2 considered the case $b = \frac{1}{2}$.)



(b) If $f(z) = (\exp(-b \operatorname{Log}_{3\pi/2}(z)))/\left(z^2+1\right)$, then by the Residue Theorem and the g/h Rule with $g(z) = \exp(-b \operatorname{Log}_{3\pi/2}(z))$, $h(z) = z^2 + 1$ and $h'(z) = 2z$,

$$\begin{aligned} I &= 2\pi i \times \operatorname{Res}(f, i) \\ &= 2\pi i \times \frac{\exp(-b \operatorname{Log}_{3\pi/2}(i))}{2i} \\ &= \pi \exp(-b(\log_e |i| + i \operatorname{Arg}_{3\pi/2}(i))) \\ &= \pi \exp(-ib\pi/2). \end{aligned}$$

(c) Splitting up the integral gives

$$\begin{aligned} I &= \int_{\Gamma_1} \cdots dz + \int_{\Gamma_2} \cdots dz + \int_{\Gamma_3} \cdots dz + \int_{\Gamma_4} \cdots dz \\ &= \pi e^{-ib\pi/2}, \end{aligned} \quad (5)$$

where

$$\begin{aligned} \int_{\Gamma_1} \cdots dz &= \int_\varepsilon^r \frac{\exp(-b \log_e t)}{t^2+1} dt \\ &= \int_\varepsilon^r \frac{t^{-b}}{t^2+1} dt \end{aligned}$$

and

$$\begin{aligned} \int_{\Gamma_3} \cdots dz &= \int_{-r}^{-\varepsilon} \frac{\exp(-b(\log_e |t| + i\pi)}{t^2+1} dt \\ &= e^{-ib\pi} \int_{-r}^{-\varepsilon} \frac{|t|^{-b}}{t^2+1} dt \\ &= e^{-ib\pi} \int_\varepsilon^r \frac{t^{-b}}{t^2+1} dt, \end{aligned}$$

since the integrand is an even function.

(d) We now estimate the integrals along Γ_2 and Γ_4.

On Γ_2, $|z| = r$ so, by the Triangle Inequality,

$$|z^2 + 1| \geq r^2 - 1.$$

Also

$$\begin{aligned} |\exp(-b \operatorname{Log}_{3\pi/2}(z))| &= \exp(-b \log_e |z|) \\ &= |z|^{-b} = r^{-b}, \quad \text{for } z \in \Gamma_2. \end{aligned}$$

Hence, by the Estimation Theorem,

$$\begin{aligned} \left| \int_{\Gamma_2} \frac{\exp(-b \operatorname{Log}_{3\pi/2}(z))}{z^2+1} dz \right| &\leq \frac{r^{-b}}{r^2-1} \times \pi r \\ &= \frac{\pi r^a}{r^2 - 1}, \quad \text{for } r > 1. \quad (6) \end{aligned}$$

Similarly

$$\begin{aligned} \left| \int_{\Gamma_4} \frac{\exp(-b \operatorname{Log}_{3\pi/2}(z))}{z^2+1} dz \right| &\leq \frac{\varepsilon^{-b}}{1 - \varepsilon^2} \times \pi\varepsilon \\ &= \frac{\pi \varepsilon^a}{1 - \varepsilon^2}, \quad \text{for } 0 < \varepsilon < 1. (7) \end{aligned}$$

(e) We now let $r \to \infty$ and $\varepsilon \to 0$ in Equation (5). Then

$$\int_{\Gamma_1} \cdots\, dz = \int_\varepsilon^r \frac{t^{-b}}{t^2+1}\,dt \to \int_0^\infty \frac{t^{-b}}{t^2+1}\,dt$$

and

$$\int_{\Gamma_3} \cdots\, dz = \mathrm{e}^{-ib\pi}\int_\varepsilon^r \frac{t^{-b}}{t^2+1}\,dt \to \mathrm{e}^{-ib\pi}\int_0^\infty \frac{t^{-b}}{t^2+1}\,dt,$$

provided that these limits exist. Their existence follows because

$$\int_{\Gamma_2} \frac{\exp(-b\operatorname{Log}_{3\pi/2}(z))}{z^2+1}\,dz \to 0 \text{ as } r \to \infty$$

(by Inequality (6)) and

$$\int_{\Gamma_4} \frac{\exp(-b\operatorname{Log}_{3\pi/2}(z))}{z^2+1}\,dz \to 0 \text{ as } \varepsilon \to 0$$

(by Inequality (7)).

Since $b = 1-a$, Equation (5) becomes,

$$\int_0^\infty \frac{t^{a-1}}{t^2+1}\,dt + \mathrm{e}^{i(a-1)\pi}\int_0^\infty \frac{t^{a-1}}{t^2+1}\,dt = \pi e^{i(a-1)\pi/2}.$$

Hence

$$\begin{aligned}\int_0^\infty \frac{t^{a-1}}{t^2+1}\,dt &= \frac{\pi e^{i(a-1)\pi/2}}{1+\mathrm{e}^{i(a-1)\pi}}\\ &= \frac{\pi i e^{ia\pi/2}}{e^{ia\pi}-1}\\ &= \frac{\pi i}{e^{ia\pi/2}-\mathrm{e}^{-ia\pi/2}}\\ &= \frac{\pi}{2\sin(a\pi/2)}.\end{aligned}$$

1.7 The improper integral

$$\int_0^\infty \frac{t^a}{t^2-t}\,dt, \quad \text{where } 0 < a < 1,$$

satisfies the hypotheses of Theorem 1.1 with $p(z) = 1$ and $q(z) = z^2 - z$: the degree of q exceeds that of p by 2, and the poles of p/q on the non-negative real axis are simple ones, at 0 and 1.

Since q has no zeros in $\mathbb{C}_{2\pi}$, we have $S = 0$. The only zero of q on the positive real axis is a simple zero at 1, which gives rise to a simple pole of

$$f_2(z) = \frac{\exp(a\operatorname{Log} z)}{z^2-z}$$

at 1. Hence

$$T = \operatorname{Res}(f_2, 1) = \frac{\exp(a\operatorname{Log} 1)}{2\cdot 1 - 1} = 1 \quad (g/h \text{ Rule}).$$

Thus, by Theorem 1.1,

$$\int_0^\infty \frac{t^a}{t^2-t}\,dt = -\pi\cot\pi a.$$

Section 2

2.1 The function $f(z) = \sum_{n=0}^{\infty} z^n$ is such that

$$f(z) = \frac{1}{1-z}, \quad \text{for } |z| < 1.$$

Hence if $\alpha \in D = \{z : |z| < 1\}$, then we have

$$f^{(n)}(\alpha) = \frac{n!}{(1-\alpha)^{n+1}}, \quad \text{for } n = 0, 1, 2, \ldots,$$

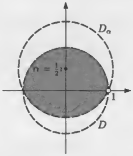
so that the Taylor series about α for f is

$$\sum_{n=0}^{\infty} \frac{(z-\alpha)^n}{(1-\alpha)^{n+1}}.$$

By the Ratio Test, the radius of convergence of this (geometric) series is $|1-\alpha|$ and so its disc of convergence is $D_\alpha = \{z : |z-\alpha| < |1-\alpha|\}$. Since $D_\alpha \cap D \neq \varnothing$, for $\alpha \in D$ (see the figure, with $\alpha = \frac{1}{2}i$), the function

$$g(z) = \sum_{n=0}^{\infty} \frac{(z-\alpha)^n}{(1-\alpha)^{n+1}} \quad (|z-\alpha| < |1-\alpha|)$$

provides a direct analytic continuation of f by Taylor series.



2.2 (a) Since

$$\begin{aligned}\operatorname{Log}_\pi(z) &= \operatorname{Log}_{2\pi}(z), &&\text{for } \operatorname{Im} z > 0,\\ \operatorname{Log}_{2\pi}(z) &= \operatorname{Log}_{3\pi}(z), &&\text{for } \operatorname{Im} z < 0,\end{aligned}$$

we deduce that

$$\begin{aligned}f_1(z) &= f_2(z), &&\text{for } \operatorname{Im} z > 0,\\ f_2(z) &= f_3(z), &&\text{for } \operatorname{Im} z < 0.\end{aligned}$$

Since the region $\{z : \operatorname{Im} z > 0\} \subseteq \mathbb{C}_\pi \cap \mathbb{C}_{2\pi}$ and the region $\{z : \operatorname{Im} z < 0\} \subseteq \mathbb{C}_{2\pi} \cap \mathbb{C}_{3\pi}$, it follows that, for $k = 1, 2$, $(f_{k+1}, \mathbb{C}_{(k+1)\pi})$ is a direct analytic continuation of $(f_k, \mathbb{C}_{k\pi})$. Also $\mathbb{C}_\pi = \mathbb{C}_{3\pi}$.

Thus $(f_1, \mathbb{C}_\pi), (f_2, \mathbb{C}_{2\pi}), (f_3, \mathbb{C}_{3\pi})$ form a closed chain.

Now, for $z \in \mathbb{C}_\pi = \mathbb{C}_{3\pi}$,

$$\begin{aligned}f_3(z) &= \exp(\tfrac{1}{2}\operatorname{Log}_{3\pi}(z))\\ &= \exp(\tfrac{1}{2}(\operatorname{Log}_\pi(z) + 2\pi i))\\ &= \mathrm{e}^{\pi i}\exp(\tfrac{1}{2}(\operatorname{Log}_\pi(z))\\ &= -f_1(z).\end{aligned}$$

Thus $f_1 \neq f_3$.

(b) Since

$$\begin{aligned}\operatorname{Log}_\pi(z) &= \operatorname{Log}_{2\pi}(z), &&\text{for } \operatorname{Im} z > 0,\\ \operatorname{Log}_{2\pi}(z) &= \operatorname{Log}_{3\pi}(z), &&\text{for } \operatorname{Im} z < 0,\\ \operatorname{Log}_{3\pi}(z) &= \operatorname{Log}_{4\pi}(z), &&\text{for } \operatorname{Im} z > 0,\\ \operatorname{Log}_{4\pi}(z) &= \operatorname{Log}_{5\pi}(z), &&\text{for } \operatorname{Im} z < 0,\end{aligned}$$

we deduce that

$$\begin{aligned}f_1(z) &= f_2(z), &&\text{for } \operatorname{Im} z > 0,\\ f_2(z) &= f_3(z), &&\text{for } \operatorname{Im} z < 0,\\ f_3(z) &= f_4(z), &&\text{for } \operatorname{Im} z > 0,\\ f_4(z) &= f_5(z), &&\text{for } \operatorname{Im} z < 0.\end{aligned}$$

Hence, as in part (a), for $k = 1, 2, 3, 4$, $(f_{k+1}, \mathbb{C}_{(k+1)\pi})$ is a direct analytic continuation of $(f_k, \mathbb{C}_{k\pi})$.

Also $\mathbb{C}_\pi = \mathbb{C}_{5\pi}$. Thus

$$(f_1, \mathbb{C}_\pi), (f_2, \mathbb{C}_{2\pi})\ (f_3, \mathbb{C}_{3\pi}), (f_4, \mathbb{C}_{4\pi}), (f_5, \mathbb{C}_{5\pi})$$

form a closed chain.

Now, for $z \in \mathbb{C}_\pi = \mathbb{C}_{5\pi}$,

$$\begin{aligned}f_5(z) &= \exp\left(\tfrac{1}{2}\operatorname{Log}_{5\pi}(z)\right)\\ &= \exp\left(\tfrac{1}{2}(\operatorname{Log}_\pi(z) + 4\pi i)\right)\\ &= \mathrm{e}^{2\pi i}\exp\left(\tfrac{1}{2}\operatorname{Log}_\pi(z)\right)\\ &= f_1(z).\end{aligned}$$

Thus $f_1 = f_5$.

2.3 First note that f and g cannot be *direct* analytic continuations of each other since $D_0 \cap D_2 = \varnothing$.

Let

$$h(z) = \frac{1}{1-z} \quad (z \in \mathbb{C} - \{1\}).$$

Then, from Example 1.1, $(h, \mathbb{C} - \{1\})$ is a direct analytic continuation of (g, D_2).

Also, since $\sum_{n=0}^{\infty} z^n$ is a geometric series with sum $1/(1-z)$, $(h, \mathbb{C} - \{1\})$ is a direct analytic continuation of (f, D_0).

Thus $(f, D_0), (h, \mathbb{C} - \{1\}), (g, D_2)$ form a chain, in which $D_0 \cap D_2 = \varnothing$, and so

$$f(z) = \sum_{n=0}^{\infty} z^n \quad (z \in D_0)$$

and

$$g(z) = \sum_{n=0}^{\infty} (-1)^{n+1}(z-2)^n \quad (z \in D_2)$$

are indirect analytic continuations of each other.

Section 3

3.1 Since $z^n \to 0$ as $n \to \infty$ for $|z| < 1$,

$$f_n(z) = \frac{1}{1+z^n} \to \frac{1}{1+0} = 1 \quad \text{as } n \to \infty,$$

for $|z| < 1$. Hence $\{f_n\}$ converges pointwise to the function $f(z) = 1$ on $E = \{z : |z| \le r\}$, where $0 < r < 1$.

Also, for $|z| \le r$,

$$\begin{aligned} |f_n(z) - f(z)| &= \left| \frac{1}{1+z^n} - 1 \right| \\ &= \left| \frac{-z^n}{1+z^n} \right| \\ &\le \frac{|z|^n}{1-|z|^n} \quad \text{(Triangle Inequality)} \\ &\le \frac{r^n}{1-r^n}. \end{aligned}$$

Since $a_n = r^n/(1-r^n)$, $n = 1, 2, \ldots$, is a null sequence of positive terms, we deduce, by the strategy, that $\{f_n\}$ converges uniformly to the function $f(z) = 1$ on E.

3.2 The power series $\sum_{n=0}^{\infty} a_n z^n$ is of the form $\sum_{n=0}^{\infty} \phi_n(z)$, where

$$\phi_n(z) = a_n z^n, \quad \text{for } n = 0, 1, 2, \ldots,$$

and, if $E = \{z : |z| \le r\}$, where $0 < r < R$, then

$$|\phi_n(z)| = |a_n||z|^n \le |a_n| r^n, \quad \text{for } z \in E.$$

Hence Assumption 1 of the M-test holds with $M_n = |a_n| r^n$. We now use the hint to show that Assumption 2 of the M-test holds. Since $r < R$, the power series $\sum_{n=0}^{\infty} a_n z^n$ is absolutely convergent at r; that is,

$$\sum_{n=0}^{\infty} M_n, \quad \text{where } M_n = |a_n| r^n,$$

is convergent. Hence Assumption 2 holds.

Thus, by the M-test, the power series $\sum_{n=0}^{\infty} a_n z^n$ is uniformly convergent on E.

3.3 Here

$$\phi_n(z) = \frac{z^n}{n^2}, \quad \text{for } n = 1, 2, \ldots,$$

and

$$\begin{aligned} |\phi_n(z)| &= \frac{|z|^n}{n^2} \\ &\le \frac{1}{n^2}, \quad \text{for } z \in E = \{z : |z| \le 1\}. \end{aligned}$$

Hence Assumption 1 of the M-test holds with $M_n = 1/n^2, n = 1, 2, \ldots$. Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, Assumption 2 of the M-test also holds.

Thus, by the M-test, the series $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ is uniformly convergent on E.

3.4 The power series $\sum_{n=0}^{\infty} a_n z^n$ is uniformly convergent on each closed disc $\{z : |z| \le r\}$, where $0 < r < R$, by Problem 3.2. Since the terms of the power series are analytic functions, it follows, by Weierstrass' Theorem, that the sum function f of the power series is analytic on $\{z : |z| < R\}$. Also the derivative f' can be obtained by term by term differentiation:

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}, \quad \text{for } |z| < R.$$

3.5 In Example 3.4 we proved that the zeta function ζ is analytic on $\mathcal{R} = \{z : \operatorname{Re} z > 1\}$, and that

$$\zeta'(z) = -\sum_{n=2}^{\infty} \frac{\log_e n}{n^z}, \quad \text{for } \operatorname{Re} z > 1. \qquad (1)$$

The formula for $\zeta''(z)$ may be obtained by term by term differentiation of Equation (1). (See Remark 3 following Weierstrass' Theorem and Remark 1 following Example 3.4.) Thus

$$\begin{aligned} \zeta''(z) &= \frac{d}{dz}\zeta'(z) \\ &= \frac{d}{dz}\left(-\sum_{n=2}^{\infty} \frac{\log_e n}{n^z} \right) \\ &= -\sum_{n=2}^{\infty} \frac{d}{dz}\left(\log_e n \, e^{-z \log_e n} \right) \\ &= -\sum_{n=2}^{\infty} -(\log_e n)^2 e^{-z \log_e n} \\ &= \sum_{n=2}^{\infty} \frac{(\log_e n)^2}{n^z}, \quad \text{for } \operatorname{Re} z > 1. \end{aligned}$$

Section 4

4.1 We put $K(z,t) = e^{-zt}t^2$, and let $D = \{z : |z| < r\}$, where $r > 0$. Then

1. K is analytic on D as a function of z, for each $t \in [-1, 1]$, and

$$\frac{\partial K}{\partial z}(z,t) = -e^{-zt}t^3;$$

2. K and $\partial K/\partial z$ are continuous on $[-1, 1]$ as functions of t, for each $z \in D$;

3. for $|z| \leq r$ and $t \in [-1,1]$, we have

$$\begin{aligned} |K(z,t)| &= \left|e^{-zt}t^2\right| \\ &\leq e^{|-zt|}t^2 \\ &\leq e^r. \end{aligned}$$

Hence, by Theorem 4.1 with $\mathcal{R} = D$, f is analytic on D, and

$$\begin{aligned} f'(z) &= \int_{-1}^{1} \frac{\partial K}{\partial z}(z,t)\,dt \\ &= -\int_{-1}^{1} e^{-zt}t^3\,dt, \quad \text{for } |z| < r. \end{aligned}$$

Since $r > 0$ was an arbitrary positive number, the result follows.

4.2 (a) We use the substitution

$$t = nu, dt = n\,du, \quad \text{for } n = 1, 2, \ldots.$$

Then, since

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\,dt, \quad \text{for } \operatorname{Re} z > 1,$$

we have

$$\begin{aligned} \Gamma(z) &= \int_0^\infty e^{-nu}(nu)^{z-1}n\,du \\ &= n^z \int_0^\infty e^{-nu}u^{z-1}\,du \quad (\text{since } n > 0, u > 0) \\ &\qquad\qquad \text{for } n = 1, 2, \ldots, \text{ and } \operatorname{Re} z > 1. \end{aligned}$$

The result follows on replacing the integration variable u by t.

(b) We use the substitution

$$t = -\log_e u, dt = -\frac{1}{u}du.$$

Then, since

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\,dt, \quad \text{for } \operatorname{Re} z > 1,$$

we have

$$\begin{aligned} \Gamma(z) &= \int_1^0 e^{\log_e u}(-\log_e u)^{z-1}\left(-\frac{1}{u}\right)du \\ &= \int_0^1 u\left(\log_e \frac{1}{u}\right)^{z-1}\frac{1}{u}\,du \\ &= \int_0^1 \left(\log_e \frac{1}{u}\right)^{z-1}du, \quad \text{for } \operatorname{Re} z > 1. \end{aligned}$$

The result follows on replacing the integration variable u by t.

4.3 (a) By the functional equation of the gamma function and Theorem 4.4, we have

$$\begin{aligned} \Gamma\left(\tfrac{5}{2}\right) &= \tfrac{3}{2}\Gamma\left(\tfrac{3}{2}\right) \\ &= \tfrac{3}{2} \cdot \tfrac{1}{2}\sqrt{\pi} = \tfrac{3}{4}\sqrt{\pi}. \end{aligned}$$

(b) By the functional equation of the gamma function and Equation (4.12), we have

$$\begin{aligned} \Gamma\left(-\tfrac{1}{2}\right) &= \frac{\Gamma\left(\frac{1}{2}\right)}{-\frac{1}{2}} \\ &= \frac{\sqrt{\pi}}{-\frac{1}{2}} = -2\sqrt{\pi}. \end{aligned}$$

SOLUTIONS TO THE EXERCISES

Section 1

1.1 (a) First note that the power series defining f is a geometric series with common ratio $z+1$, so that, for $|z+1|<1$,

$$f(z) = \frac{z+1}{1-(z+1)}$$
$$= -1 - \frac{1}{z}.$$

Thus if $\mathcal{R} = \{z : |z+1| < 1\}$ and $\mathcal{S} = \mathbb{C} - \{0\}$, then f and g agree on the region $\mathcal{R} \subseteq \mathcal{R} \cap \mathcal{S}$. Hence f and g are direct analytic continuations of each other.

(b) Since

$$\operatorname{Log}(1+w) = w - \tfrac{1}{2}w^2 + \tfrac{1}{3}w^3 - \cdots, \quad \text{for } |w|<1,$$

we deduce, on substituting $w = -(z+1)$, that

$$\operatorname{Log}(-z) = -(z+1) - \tfrac{1}{2}(z+1)^2 - \tfrac{1}{3}(z+1)^3 - \cdots, \quad \text{for } |z+1|<1.$$

Hence, for $|z+1|<1$,

$$f(z) = -\operatorname{Log}(-z),$$
$$= -(\log_e |z| + i\operatorname{Arg}(-z)).$$

Now, for $|z+1|<1$, the point $-z$ lies in the right half-plane and so

$$\operatorname{Arg}(-z) + \pi = \operatorname{Arg}_{2\pi}(z).$$

Hence, for $|z+1|<1$,

$$f(z) = -(\log_e |z| + i\operatorname{Arg}_{2\pi}(z) - i\pi)$$
$$= i\pi - \operatorname{Log}_{2\pi}(z).$$

Thus if $\mathcal{R} = \{z : |z+1| < 1\}$ and $\mathcal{S} = \mathbb{C}_{2\pi}$, then f and g agree on the region $\mathcal{R} \subseteq \mathcal{R} \cap \mathcal{S}$. Hence f and g are direct analytic continuations of each other.

1.2 (a) Since the given power series has disc of convergence $\{z : |z| < 1\}$, we deduce, by the Differentiation Rule, that

$$f'(z) = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n}, \quad \text{for } |z|<1, \tag{1}$$

and hence that

$$zf'(z) = \sum_{n=1}^{\infty} \frac{z^n}{n}, \quad \text{for } |z|<1,$$
$$= -\operatorname{Log}(1-z).$$

Thus

$$f'(z) = \frac{-\operatorname{Log}(1-z)}{z}, \quad \text{for } 0<|z|<1,$$

as required.

(b) The function

$$g(z) = \frac{-\operatorname{Log}(1-z)}{z} \quad (z \in \mathbb{C} - \{x \in \mathbb{R} : x = 0 \text{ or } x \geq 1\})$$

has a removable singularity at 0, which can be removed by putting $g(0) = f'(0) = 1$ (from Equation (1)). Thus g and f' agree on $\{z : |z| < 1\}$ and so g is a direct analytic continuation from $\{z : |z| < 1\}$ to the domain of g, which is $\mathbb{C} - \{x \in \mathbb{R} : x \geq 1\}$.

(c) Since $\mathbb{C} - \{x \in \mathbb{R} : x \geq 1\}$ is simply-connected, g has a primitive G on this region by the Primitive Theorem and, by adding a constant if necessary, we may arrange that $G(0) = f(0) = 0$. Then both G and f are primitives of f' on $\{z : |z| < 1\}$ which agree at 0. Hence

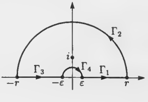
$$G(z) = f(z), \quad \text{for } |z|<1,$$

so that G is a direct analytic continuation of f from $\{z : |z| < 1\}$ to $\mathbb{C} - \{x \in \mathbb{R} : x \geq 1\}$.

1.3 Consider the contour integral

$$I = \int_\Gamma \frac{\operatorname{Log}_{3\pi/2}(z)}{(1+z^2)^2}\,dz$$

where $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$, as shown in the figure.



If $f(z) = (\operatorname{Log}_{3\pi/2}(z))/(1+z^2)^2$, then, by the Residue Theorem and the higher-order formula for residues (*Unit C1*, Theorem 1.2),

$$I = 2\pi i \times \operatorname{Res}(f, i)$$
$$= 2\pi i \times \lim_{z\to i}\left(\frac{d}{dz}\frac{(z-i)^2 \operatorname{Log}_{3\pi/2}(z)}{(z^2+1)^2}\right)$$
$$= 2\pi i \times \lim_{z\to i}\left(\frac{d}{dz}\frac{\operatorname{Log}_{3\pi/2}(z)}{(z+i)^2}\right)$$
$$= 2\pi i \times \lim_{z\to i}\left(\frac{(z+i)^2(1/z) - 2(z+i)\operatorname{Log}_{3\pi/2}(z)}{(z+i)^4}\right)$$
$$= 2\pi i \times \left(\frac{(2i)^2/i - 4i \times i\pi/2}{(2i)^4}\right)$$
$$= \frac{\pi}{2}\left(\frac{i\pi}{2} - 1\right).$$

Splitting up the integral gives

$$I = \int_{\Gamma_1} \cdots\,dz + \int_{\Gamma_2} \cdots\,dz + \int_{\Gamma_3} \cdots\,dz + \int_{\Gamma_4} \cdots\,dz. \tag{2}$$

Now

$$\int_{\Gamma_1} \cdots\,dz = \int_\varepsilon^r \frac{\log_e t}{(1+t^2)^2}\,dt$$

and

$$\int_{\Gamma_3} \cdots\,dz = \int_{-r}^{-\varepsilon} \frac{\log_e |t| + i\pi}{(1+t^2)^2}\,dt$$
$$= \int_\varepsilon^r \frac{\log_e |t|}{(1+t^2)^2}\,dt + i\pi \int_\varepsilon^r \frac{1}{(1+t^2)^2}\,dt,$$

since the integrand is an even function.

By the Estimation Theorem and the Triangle Inequality, and the fact that

$$\left|\operatorname{Log}_{3\pi/2}(z)\right| = \left|\log_e |z| + i\operatorname{Arg}_{3\pi/2}(z)\right|$$
$$\leq \left|\log_e |z|\right| + \left|\operatorname{Arg}_{3\pi/2}(z)\right|$$
$$\leq \log_e r + \pi, \quad \text{for } z \in \Gamma_2,$$

we have

$$\left|\int_{\Gamma_2} \frac{\operatorname{Log}_{3\pi/2}(z)}{(1+z^2)^2}\,dz\right| \leq \frac{\log_e r + \pi}{(r^2-1)^2} \times \pi r, \quad \text{for } r>1, \tag{3}$$

and similarly

$$\left|\int_{\Gamma_4} \frac{\operatorname{Log}_{3\pi/2}(z)}{(1+z^2)^2}\,dz\right| \leq \frac{\pi - \log_e \varepsilon}{(1-\varepsilon^2)^2} \times \pi\varepsilon, \quad \text{for } 0<\varepsilon<1. \tag{4}$$

We now let $r \to \infty$ and $\varepsilon \to 0$ in Equation (2). Then

$$\int_\varepsilon^r \frac{\log_e t}{(1+t^2)^2}\,dt \to \int_0^\infty \frac{\log_e t}{(1+t^2)^2}\,dt$$

and

$$\int_\varepsilon^r \frac{1}{(1+t^2)^2}\,dt \to \int_0^\infty \frac{1}{(1+t^2)^2}\,dt,$$

provided that these limits exist. Their existence follows because

$$\int_{\Gamma_2} \frac{\mathrm{Log}_{3\pi/2}(z)}{(1+z^2)^2}\,dz \to 0 \quad \text{(by Inequality(3))}$$

and

$$\int_{\Gamma_4} \frac{\mathrm{Log}_{3\pi/2}(z)}{(1+z^2)^2}\,dz \to 0 \quad \text{(by Inequality (4))}.$$

Equation (2) becomes

$$2\int_0^\infty \frac{\log_e t}{(1+t^2)^2}\,dt + i\pi\int_0^\infty \frac{1}{(1+t^2)^2}\,dt = \frac{\pi}{2}\left(i\frac{\pi}{2}-1\right).$$

Hence, equating real parts, we obtain

$$\int_0^\infty \frac{\log_e t}{(1+t^2)^2}\,dt = -\frac{\pi}{4}.$$

(As a check, notice that equating imaginary parts gives

$$\int_0^\infty \frac{1}{(1+t^2)^2}\,dt = \frac{\pi}{4},$$

which can be verified by substituting $t = \tan\theta$, $dt = \sec^2\theta\,d\theta$:

$$\begin{aligned}\int_0^\infty \frac{1}{(1+t^2)^2}\,dt &= \int_0^{\pi/2} \frac{\sec^2\theta}{(1+\tan^2\theta)^2}\,d\theta\\ &= \int_0^{\pi/2} \cos^2\theta\,d\theta\\ &= \int_0^{\pi/2} \tfrac{1}{2}(\cos 2\theta + 1)\,d\theta = \frac{\pi}{4}.)\end{aligned}$$

1.4 We apply Theorem 1.1 with $a = \frac{1}{2}, p(z) = z$ and $q(z) = (z^2+1)(z-1)$. This is permissible since the degree of q exceeds that of p by 2 and the only pole of p/q on the non-negative real axis is a simple one, at 1.

The only singularities of p/q in $\mathbb{C}_{2\pi}$ are simple poles at i and $-i$, and the only singularity of p/q on the positive real axis is a simple pole at 1. Hence, using the g/h Rule with

$$g_1(z) = z\exp\left(\tfrac{1}{2}\mathrm{Log}_{2\pi}(z)\right), \quad g_2(z) = z\exp\left(\tfrac{1}{2}\mathrm{Log}\, z\right),$$

$$h(z) = q(z) = (z^2+1)(z-1), \quad h'(z) = 3z^2 - 2z + 1,$$

we obtain

$$\begin{aligned}\mathrm{Res}(f_1, i) &= \frac{i \times \exp\left(\frac{1}{2}\mathrm{Log}_{2\pi}(i)\right)}{3\times i^2 - 2\times i + 1}\\ &= \frac{ie^{i\pi/4}}{-2-2i} = -\frac{i}{2\sqrt{2}},\end{aligned}$$

$$\begin{aligned}\mathrm{Res}(f_1, -i) &= \frac{-i \times \exp\left(\frac{1}{2}\mathrm{Log}_{2\pi}(-i)\right)}{3\times(-i)^2 - 2\times(-i) + 1}\\ &= \frac{-ie^{i3\pi/4}}{-2+2i} = -\frac{i}{2\sqrt{2}},\end{aligned}$$

$$\begin{aligned}\mathrm{Res}(f_2, 1) &= \frac{1 \times \exp\left(\frac{1}{2}\mathrm{Log}\,1\right)}{3\times 1^2 - 2\times 1 + 1}\\ &= \tfrac{1}{2}.\end{aligned}$$

Hence, by Theorem 1.1 with $a = \frac{1}{2}$,

$$\begin{aligned}\int_0^\infty \frac{t^{3/2}}{(t^2+1)(t-1)}\,dt &= \int_0^\infty \frac{t}{(t^2+1)(t-1)}t^{1/2}dt\\ &= -\left(\pi e^{-\pi i/2}\,\mathrm{cosec}\,\tfrac{1}{2}\pi\right)S\\ &\quad -\left(\pi\cot\tfrac{1}{2}\pi\right)T\\ &= \pi i\left(-\frac{i}{2\sqrt{2}} - \frac{i}{2\sqrt{2}}\right)\\ &= \frac{\pi}{\sqrt{2}}.\end{aligned}$$

Remark Since $a = \frac{1}{2}, \pi\cot\pi a = 0$ and so we did not need to evaluate $T = \mathrm{Res}(f_2, 1)$.

Section 2

2.1 First note that f and g cannot be *direct* analytic continuations of each other since the intersection of their domains is empty. Now

$$f(z) = \frac{1}{1-z}, \quad \text{for } |z| < 1,$$

and

$$\begin{aligned}g(z) &= -\frac{1/z}{1-1/z}\\ &= \frac{1}{1-z}, \quad \text{for } |z| > 1.\end{aligned}$$

Thus, if

$$h(z) = \frac{1}{1-z} \quad (z \in \mathbb{C} - \{1\}),$$

then $f = h$ on the region $\mathcal{T}_1 = \{z : |z| < 1\} \subseteq \mathcal{T}_1 \cap (\mathbb{C} - \{1\})$ and $g = h$ on the region $\mathcal{T}_2 = \{z : |z| > 1\} \subseteq \mathcal{T}_2 \cap (\mathbb{C} - \{1\})$. So

$$(f, \{z : |z| < 1\}), (h, \mathbb{C} - \{1\}), (g, \{z : |z| > 1\})$$

form a chain. Hence f and g are indirect analytic continuations of each other.

2.2 First note that f and g cannot be *direct* analytic continuations of each other since the intersection of their domains is empty. Now if $z \in A = \{z : |z-1| < 1\}$, then

$$\begin{aligned}f(z) &= \mathrm{Log}(1 + (z-1))\\ &= \mathrm{Log}\, z,\end{aligned}$$

and, by Exercise 1.1(b), if $z \in B = \{z : |z+1| < 1\}$, then

$$\begin{aligned}g(z) &= i\pi + \mathrm{Log}(-z)\\ &= i\pi + (\mathrm{Log}_{2\pi}(z) - i\pi)\\ &= \mathrm{Log}_{2\pi}(z).\end{aligned}$$

Thus (f, A) and $(\mathrm{Log}, \mathbb{C}_\pi)$ agree on the region $A \subseteq A \cap \mathbb{C}_\pi$, and (g, B) and $(\mathrm{Log}_{2\pi}, \mathbb{C}_{2\pi})$ agree on the region $B \subseteq B \cap \mathbb{C}_{2\pi}$. Hence (f, A) and $(\mathrm{Log}, \mathbb{C}_\pi)$ are direct analytic continuations of each other, and so are (g, B) and $(\mathrm{Log}_{2\pi}, \mathbb{C}_{2\pi})$.

Thus if $f_1(z) = \mathrm{Log}\, z$ and $g_1(z) = \mathrm{Log}_{2\pi}(z)$, then, since f_1 and g_1 are direct analytic continuations of each other, the functions

$$(f, \{z : |z-1| < 1\}), (f_1, \mathbb{C}_\pi), (g_1, \mathbb{C}_{2\pi}), (g, \{z : |z+1| < 1\})$$

form a chain. Hence f and g are indirect analytic continuations of each other.

(Actually, we only need $(f_1, \mathbb{C}_\pi)$ between f and g because f_1 agrees with g_1 on the upper open half-plane and hence agrees with g on an open subset of $\{z : |z+1| < 1\}$.)

Section 3

3.1 We use the strategy for proving uniform convergence. First note that, for any $z \in \mathbb{C}$,

$$f_n(z) = z + \frac{z^2}{n} \to z + 0 = z \quad \text{as } n \to \infty.$$

Hence f_n converges pointwise to the limit function $f(z) = z$ on $\mathbb{C}$, and so on any closed disc in $\mathbb{C}$.

Now let $E = \{z : |z| \le r\}$, for some $r > 0$. Then

$$\begin{aligned} |f_n(z) - f(z)| &= \left| z + \frac{z^2}{n} - z \right| \\ &= \frac{|z|^2}{n} \\ &\le \frac{r^2}{n}, \quad \text{for } n = 1, 2, \ldots, \text{ and all } z \in E. \end{aligned}$$

Since $\{r^2/n\}$ is a null sequence for each fixed $r > 0$, we deduce that $\{f_n\}$ converges uniformly to f on E.

3.2 We apply the M-test. Here

$$\phi_n(z) = \frac{z^n}{1+z^n}, \quad n = 1, 2, \ldots,$$

and, for $z \in E = \{z : |z| \le r\}$, where $0 < r < 1$,

$$\begin{aligned} |\phi_n(z)| &\le \frac{|z|^n}{1 - |z|^n} \quad \text{(by the Triangle Inequality)} \\ &\le \frac{r^n}{1 - r^n}, \quad \text{for } z \in E, \end{aligned}$$

since $|z|^n \le r^n \implies 1 - |z|^n \ge 1 - r^n$.

Hence Assumption 1 of the M-test holds with $M_n = r^n/(1 - r^n)$. Since

$$\frac{r^n}{1 - r^n} \le \frac{r^n}{1 - r} \quad (r^n \le r, \text{ for } n = 1, 2, \ldots),$$

and $\sum_{n=1}^{\infty} r^n$ is convergent (because $0 < r < 1$),

we deduce, by the Comparison Test, that $\sum_{n=1}^{\infty} M_n$ is convergent.

Thus Assumption 2 of the M-test also holds and so

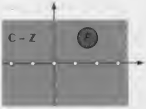
$$\sum_{n=1}^{\infty} \frac{z^n}{1+z^n}$$

is uniformly convergent on each closed disc $E = \{z : |z| \le r\}$, for $0 < r < 1$, and hence on each closed disc in $\{z : |z| < 1\}$.

It follows from Weierstrass' Theorem that the sum function f is analytic on the open unit disc $\{z : |z| < 1\}$. Furthermore, we can obtain the derivative of f by term by term differentiation of the series:

$$\begin{aligned} f'(z) &= \sum_{n=1}^{\infty} \frac{d}{dz}\left(\frac{z^n}{1+z^n}\right) \\ &= \sum_{n=1}^{\infty} \frac{nz^{n-1}}{(1+z^n)^2}, \quad \text{for } |z| < 1. \end{aligned}$$

3.3 Let E be a closed disc in $\mathbb{C} - \mathbb{Z}$ (see the figure), and choose $r > 0$ so that $E \subseteq \{z : |z| \le r\}$.



Following the hint we show that the series $\sum_{n \ge 2r} \frac{1}{z^2 - n^2}$ is uniformly convergent on E, by using the M-test with $\phi_n(z) = \frac{1}{z^2 - n^2}$, for $n \ge 2r$.

For $z \in E$ and $n \ge 2r$, we have

$$\begin{aligned} |\phi_n(z)| &= \frac{1}{|z^2 - n^2|} \\ &\le \frac{1}{n^2 - |z|^2} \quad \text{(Triangle Inequality)} \\ &\le \frac{1}{n^2 - r^2} \quad (\text{since } |z| \le r). \end{aligned}$$

Hence Assumption 1 of the M-test holds with $M_n = 1/\left(n^2 - r^2\right)$, for $n \ge 2r$.

Now note that $r \le \frac{1}{2}n$, for $n \ge 2r$, and so

$$\frac{1}{n^2 - r^2} \le \frac{1}{n^2 - \left(\frac{1}{2}n\right)^2} = \frac{4}{3n^2}, \quad \text{for } n \ge 2r.$$

Since $\sum_{n=1}^{\infty} 1/n^2$ is convergent, we deduce, by the Comparison Test, that

$$\sum_{n \ge 2r} M_n = \sum_{n \ge 2r} \frac{1}{n^2 - r^2}$$

is convergent, so that Assumption 2 of the M-test also holds. Thus $\sum_{n \ge 2r} \frac{1}{z^2 - n^2}$ is uniformly convergent on E,

and so, therefore, is $\sum_{n=1}^{\infty} \frac{1}{z^2 - n^2}$.

Hence, by Weierstrass' Theorem, the function f is analytic on the region $\mathbb{C} - \mathbb{Z}$. Furthermore, we can obtain the derivative of f by term by term differentiation of the series:

$$\begin{aligned} f'(z) &= \sum_{n=1}^{\infty} \frac{d}{dz}\left(\frac{1}{z^2 - n^2}\right) \\ &= \sum_{n=1}^{\infty} -\frac{2z}{(z^2 - n^2)^2} \\ &= -2z \sum_{n=1}^{\infty} \frac{1}{(z^2 - n^2)^2}, \quad \text{for } z \in \mathbb{C} - \mathbb{Z}. \end{aligned}$$

Remark In *Unit C1*, Exercise 4.2 we obtained the representation

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}, \quad \text{for } z \in \mathbb{C} - \mathbb{Z}.$$

Thus the function f of this exercise is in fact given by

$$f(z) = \frac{1}{2z}\left(\pi \cot \pi z - \frac{1}{z}\right), \quad \text{for } z \in \mathbb{C} - \mathbb{Z}.$$

Section 4

4.1 We put $K(z,t) = \sin(zt^2)$ and let $D = \{z : |z| < r\}$, where $r > 0$. Then

1. K is analytic on D as a function of z, for each $t \in [-1, 1]$, and
$$\frac{\partial K}{\partial z}(z,t) = t^2 \cos(zt^2);$$
2. K and $\partial K/\partial z$ are continuous on $[-1, 1]$ as functions of t, for each $z \in D$;
3. for $|z| \le r$ and $t \in [-1, 1]$, we have
$$\begin{aligned}|K(z,t)| &= \left|\sin(zt^2)\right| \\ &= \left|\frac{e^{izt^2} - e^{-izt^2}}{2i}\right| \\ &\le \tfrac{1}{2}\left(\left|e^{izt^2}\right| + \left|e^{-izt^2}\right|\right) \\ &\quad \text{(by the Triangle Inequality)} \\ &\le \tfrac{1}{2}\left(e^{|z|t^2} + e^{|z|t^2}\right) \quad (\text{since } |e^z| \le e^{|z|}) \\ &\le e^r.\end{aligned}$$

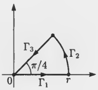
Hence, by Theorem 4.1 with $\mathcal{R} = D$, f is analytic on D, and
$$\begin{aligned}f'(z) &= \int_{-1}^{1} \frac{\partial K}{\partial z}(z,t)\, dt \\ &= \int_{-1}^{1} t^2 \cos(zt^2)\, dt, \quad \text{for } |z| < r.\end{aligned}$$
Since $r > 0$ was an arbitrary positive number, this formula holds for all $z \in \mathbb{C}$.

4.2 (a) By the functional equation of the gamma function and the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we have
$$\Gamma(-\tfrac{3}{2}) = \Gamma(-\tfrac{1}{2})/(-\tfrac{3}{2}) = \Gamma(\tfrac{1}{2})/((-\tfrac{3}{2})(-\tfrac{1}{2})) = \tfrac{4}{3}\sqrt{\pi}.$$
(b) By the functional equation of the gamma function, we have
$$\begin{aligned}\Gamma(-3+i) &= \Gamma(-2+i)/(-3+i) \\ &= \Gamma(-1+i)/((-3+i)(-2+i)) \\ &= \Gamma(i)/((-3+i)(-2+i)(-1+i)),\end{aligned}$$
so that
$$\frac{\Gamma(-3+i)}{\Gamma(i)} = \frac{1}{(-3+i)(-2+i)(-1+i)} = -\frac{i}{10}.$$

4.3 (a) By the given formula and Theorem 4.4,
$$V_1 = \frac{\pi^{1/2}}{\Gamma(3/2)} r = \frac{\pi^{1/2}}{\frac{1}{2}\pi^{1/2}} r = 2r.$$
(b) By the given formula,
$$V_2 = \frac{\pi}{\Gamma(2)} r^2 = \pi r^2.$$
(c) By the given formula and Theorem 4.4,
$$V_3 = \frac{\pi^{3/2}}{\Gamma(5/2)} r^3 = \frac{\pi^{3/2}}{\frac{3}{4}\pi^{1/2}} r^3 = \tfrac{4}{3}\pi r^3.$$
(d) By the given formula,
$$V_4 = \frac{\pi^2}{\Gamma(3)} r^4 = \tfrac{1}{2}\pi^2 r^4.$$

4.4 If $\alpha = \beta = \frac{1}{2}$, then the integral is
$$\begin{aligned}\int_0^1 t^{-1/2}(1-t)^{-1/2} dt &= \int_0^1 \frac{1}{\sqrt{t-t^2}}\, dt \\ &= \int_0^{\pi/2} \frac{2\sin\theta\cos\theta}{\sqrt{\sin^2\theta - \sin^4\theta}}\, d\theta \\ &\quad (t = \sin^2\theta, dt = 2\sin\theta\cos\theta\, d\theta) \\ &= \int_0^{\pi/2} \frac{2\sin\theta\cos\theta}{\sqrt{\sin^2\theta\cos^2\theta}}\, d\theta \\ &\quad (\cos^2\theta = 1 - \sin^2\theta) \\ &= 2\int_0^{\pi/2} 1\, d\theta = \pi.\end{aligned}$$
On the other hand,
$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2})} = \frac{(\sqrt{\pi})^2}{1} = \pi,$$
so the identity is verified for $\alpha = \beta = \frac{1}{2}$.

4.5 Since $f(z) = e^{-z^2}$ is entire, we deduce from Cauchy's Theorem that
$$I = \int_\Gamma e^{-z^2} dz = 0.$$
Now split up this integral as shown in the figure.



Then
$$\int_{\Gamma_1} e^{-z^2} dz + \int_{\Gamma_2} e^{-z^2} dz + \int_{\Gamma_3} e^{-z^2} dz = 0. \tag{1}$$
We have
$$\int_{\Gamma_1} e^{-z^2} dz = \int_0^r e^{-t^2} dt \tag{2}$$
and, using the parametrization $\gamma(t) = te^{i\pi/4}$ $(t \in [0, r])$ of $\widetilde{\Gamma}_3$,
$$\begin{aligned}\int_{\Gamma_3} e^{-z^2} dz &= -\int_{\widetilde{\Gamma}_3} e^{-z^2} dz \\ &= -\int_0^r \exp\left(-\left(te^{i\pi/4}\right)^2\right) e^{i\pi/4} dt \\ &= -e^{i\pi/4} \int_0^r e^{-it^2} dt.\end{aligned} \tag{3}$$
On Γ_2 we use the suggested parametrization
$\gamma(t) = re^{it}$ $(t \in [0, \pi/4])$:
$$\begin{aligned}\int_{\Gamma_2} e^{-z^2} dz &= \int_0^{\pi/4} \exp\left(-\left(re^{it}\right)^2\right) rie^{it} dt \\ &= ir\int_0^{\pi/4} e^{-r^2(\cos 2t + i\sin 2t)} e^{it}\, dt.\end{aligned}$$

Hence, by *Unit B1*, Lemma 4.1,

$$\left|\int_{\Gamma_2} e^{-z^2}\,dz\right| \le r\int_0^{\pi/4} e^{-r^2\cos 2t}\,dt$$

$$\le r\int_0^{\pi/4} e^{-r^2(1-4t/\pi)}\,dt \quad \text{(by the hint)}$$

$$= re^{-r^2}\int_0^{\pi/4} e^{r^2 4t/\pi}\,dt$$

$$= re^{-r^2}\left[\frac{\pi}{4r^2}e^{4r^2t/\pi}\right]_0^{\pi/4}$$

$$= \frac{\pi}{4re^{r^2}}\left(e^{r^2}-1\right)$$

$$\le \frac{\pi}{4r}.$$

Thus

$$\int_{\Gamma_2} e^{-z^2}\,dz \to 0 \quad \text{as } r \to \infty.$$

So Equations (1), (2) and (3) give

$$\int_0^\infty e^{-t^2}\,dt = e^{i\pi/4}\int_0^\infty e^{-it^2}\,dt$$

$$= e^{i\pi/4}\left(\int_0^\infty \left(\cos(t^2) - i\sin(t^2)\right)dt\right).$$

Since

$$\int_0^\infty e^{-t^2}\,dt = \tfrac{1}{2}\sqrt{\pi} \quad \text{(see the proof of Theorem 4.4)},$$

we deduce that

$$\int_0^\infty \cos(t^2)\,dt - i\int_0^\infty \sin(t^2)\,dt = \frac{1}{2}\sqrt{\pi}e^{-i\pi/4}$$

$$= \frac{1}{2}\sqrt{\pi}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right).$$

Hence

$$\int_0^\infty \cos(t^2)\,dt = \frac{1}{2}\sqrt{\frac{\pi}{2}} \quad \text{and} \quad \int_0^\infty \sin(t^2)\,dt = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$